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ON RANDOM OCCURRENCES IN SPACE AND TIME, WHEN FOLLOWED BY A CLOSED INTERVAL.

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I. *Introductory.*

The object of the present paper is to investigate a little more fully than has hitherto been done the formulae for random occurrences in space and time, and to test their adequacy on the basis of experimental data.

The fundamental formula, which was first stated and proved by Whitworth*, is obtained as follows: An event happens at random once in a period m, therefore its chance of occurring in an interval of time or space δt is $\frac{\delta t}{m}$, and of its failing to occur $\left(1-\frac{\delta t}{m}\right)$. If we now take n such intervals, its chance of not occurring in time $n\delta t$ is $\left(1-\frac{\delta t}{m}\right)^n$, accordingly if we make $t=n\delta t$ and suppose n to become indefinitely large and δt indefinitely small, we have

$$\operatorname*{Lt}_{n=\infty}\left(1-\frac{t}{nm}\right)^n = e^{-\frac{t}{m}},$$

which is therefore the chance of non-occurrence during an interval t, when the occurrence measured on a very long period averages once in the interval m.

Whitworth gives no experimental data to confirm this theory and apparently did not fully recognise its immense importance, if confirmed, for medical and sociological "events." That is to say, as a means of distinguishing between random and associated occurrences, for which it offers, with some expansions, a most valuable criterion.

At first sight it might seem an easy task to justify or refute such a law of distribution by actual experiment. But in endeavouring to find how such a series of events may be obtained by observation or experiment there are serious difficulties to be encountered. Dr M. Greenwood suggested dealing experimentally with the matter, and he drew numbered counters from a bag and considered the differences between successive numbers to be intervals between events. But such a case does not fully accord with theory, for the counters being numbered by units the intervals will be clustered in masses differing by a unit. The chief difficulty of any experimental demonstration lies in allowing for a large number of simultaneous occurrences. For example, if we take taxi-cabs passing a given lamp-post in a given street in one direction, it is perhaps only possible for one or two to pass at one

* *Choice and Chance*, Cambridge, 1901, 5th Edition, p. 200.

instant of time. If only one is able to pass, there will be a "closed-time" β, measured by the length of the taxi plus a certain margin of safety, both varying with the size of the taxi and the bump of precaution of the second driver. In considering experiments likely to exhibit the law, we were repeatedly met by the existence of such a "closed-time" β intervening between event and event. Some of the most important cases to which the theory may be applied, socially or medically, involve such a closed time between occurrences. Accordingly it seems desirable to extend the theory to cases in which there is of necessity a "closed-time" β within which no event can follow a given occurrence. As most experimental and many social and vital phenomena involve a β as well as an m, it becomes necessary to devise an experiment which will give some approach to a constant β and a constant m.

If a chronograph be running and marks seconds, and an observer taps on the occurrence of random events, he will be unable to tap any number of simultaneous events; they will be separated by at least the reaction time requisite for noting the event and tapping the key. But it is by no means certain that the β and m in such a case remain really constant; in fact it will be shown that experiments suggest that they do not. There may be slight irregularities due to the apparatus, such as in the time of swing of the pendulum or the rate at which the tape is moving, but more important than any of these will be the fluctuating "personal equation" of the observer.

A suitable series of random intervals seemed to be those between the successive occurrences of the figure "5" in the units place of columns of numbers, each containing at least four figures, in a Census Report. The unit figures were read down as far as possible at a constant rate, tapping the key of the chronograph on coming to a "5." Six half-hour readings were taken in this way and it was assumed that the values of m and β for the six tapes were not significantly different from each other, so that the whole might be added to give a homogeneous distribution. It will be shown later how far this assumption was justified.

A constant β, such as the theory postulates, would be given if the length of the interval between two "5's" were measured by the number of intervening figures. But this case would be similar to that of drawing numbered counters from a bag; the intervals could only differ by units. By introducing the personal factor we obtain a continuous distribution but the constants β and m are sure to fluctuate slightly throughout the experiment. It was hoped that this would not materially affect the results.

II. *The Frequency of Intervals in an indefinitely long Time**.

The first case to be considered is that of the frequency of intervals which elapse between two successive occurrences of the event when the period in which the observations are taken is continuous and indefinitely long. We have seen that the

* We shall use the terminology of "time" throughout, but the intervals may always be interpreted as intervals of space and the occurrences as random marks on a line.

chance of non-occurrence during an interval t, when the occurrence measured in a very long period averages once in time m, is $e^{-\frac{t}{m}}$. If there are N intervals, it follows that the frequency of intervals of length lying between t and $t+dt$ will be given by $f_n\,dt = \frac{N}{m} e^{-\frac{t}{m}}\,dt$, or the frequency curve is $y_t = \frac{N}{m} e^{-\frac{t}{m}}$. Now we have to consider the case in which an occurrence is always followed by a "closed-interval" β, so the form of the curve will be $y_t = \frac{N}{m} e^{-\frac{(t-\beta)}{m}}$

Thus, having observed the frequency of intervals in an indefinitely long time, it remains to find the values of β and m which will fit this curve best. The beginning of the range, being at the point $t=\beta$, is not known *a priori*, so that the moments of the observed distribution cannot be found in the usual way. The following method is used to determine suitable values of β and m, applying for fitting, as now usual, the method of moments.

The curve starts at β and runs theoretically to $t=\infty$. Let

$$I_{s,r} = \frac{N}{m}\int_{\beta}^{t_r} t^s\, e^{-\frac{(t-\beta)}{m}}\,dt;$$

or, integrating by parts,

$$I_{s,r} = N\{\beta^s + sm\beta^{s-1} + s(s-1)m^2\beta^{s-2} + \ldots\} - Ne^{-\frac{(t_r-\beta)}{m}}\{t_r^s + smt_r^{s-1} + s(s-1)m^2t_r^{s-2}\} \ldots\text{(i)}.$$

Accordingly:

$$I_{1,\infty} = N(\beta + m),$$

$$I_{2,\infty} = N(\beta^2 + 2m\beta + 2m^2),$$

$$I_{0,t_r} = N\left(1 - e^{-\frac{(t_r-\beta)}{m}}\right) = n_{t_r} = N\bar{n}_{t_r}, \text{ say},$$

$$I_{1,t_r} = N\{\beta + m - e^{-\frac{(t_r-\beta)}{m}}(t_r + m)\}$$
$$= N\{\beta - t_r + \bar{n}_{t_r}(t_r + m)\},$$

$$I_{2,t_r} = N\{\beta^2 + 2m\beta + 2m^2 - e^{-\frac{(t_r-\beta)}{m}}(t_r^2 + 2mt_r + 2m^2)\}$$
$$= N\{\beta^2 - t_r^2 + 2m(\beta - t_r) + \bar{n}_{t_r}(t_r^2 + 2mt_r + 2m^2)\} \ldots\ldots\text{(ii)}.$$

Now $I_{1,\infty} - I_{1,t_r}$ = 1st moment of all observations about $t=0$ excluding the frequency up to intervals t_r.

$I_{2,\infty} - I_{2,t_r}$ = 2nd moment of all observations about $t=0$, excluding the frequency up to intervals t_r.

Let these be respectively M_1 and M_2.

Then we have

$$M_1 = Ne^{-\frac{(t_r-\beta)}{m}},$$

$$M_2 = Ne^{-\frac{(t_r-\beta)}{m}}(t_r^2 + 2mt_r + 2m^2),$$

$$\frac{M_2}{M_1} = \lambda, \text{ say} = \frac{t_r^2 + 2mt_r + 2m^2}{t_r + m},$$

or
$$2(m+t_r)^2 - (\lambda + t_r)(m + t_r) + t_r^2 = 0 \quad \text{.....(iii).}$$

This is a quadratic to find $(m + t_r)$ and therefore m when λ is known.

Now the curve is exponential and we start with a finite ordinate at $t = \beta$, and therefore in dealing with M_1 and M_2 from the observations we are bound to use careful adjustments, i.e. the abruptness coefficients for the moments*. Not knowing β, however, we cannot evaluate the moments of the first group. But we do know an approximate value of β, because we can note what the least interval observed is, and we may assume t_r to be something greater than this. In actual observation work Equation (iii) did not give good results and the reason for this is fairly obvious. In determining the moments we neglect a very important factor, the moments of the first group, and throw all the weight of determining β and m on the later frequencies.

A start may be made in the following manner. Take approximate values of β and m, say β_0 and m_0, and calculate the position of the mean and the moments of the first group on the assumption that these values are correct, and add them to the observed moments of the remainder to obtain the total observed moments for equating against our theoretical results for the determination of a new β and m. In doing this we may obtain β_0 and m_0 from the area value and the first moment of all the data without the first group (observed frequency n_1). Let the limit of the first group be $t = \gamma$; then we shall take γ as close to β as is safe in our ignorance as to the exact value of β in order to make the influence of this first group as small as possible. For example, if β be about ·3, we shall take $\gamma = ·5$ rather than 1, and reckon, say, our groupings from ·5 to 1·5, 1·5 to 2·5 etc., rather than from 1 to 2, 2 to 3 etc. of our units of time.

Now suppose we find the first moment of $N - n_1$ of the observations after $t = \gamma$ (this being M_1) and also $N - n_1$. These give

$$N - n_1 = Ne^{-\frac{(\gamma-\beta_0)}{m_0}},$$

$$M_1 = Ne^{-\frac{(\gamma-\beta_0)}{m_0}}(\gamma + m_0).$$

Then
$$m_0 = \frac{M_1}{(N - n_1)} - \gamma \quad \text{.....(iv),}$$

and
$$\beta_0 = \gamma - \frac{m_0\{\log_{10} N - \log_{10}(N - n_1)\}}{\log_{10} e} \quad \text{.....(v).}$$

Thus our preliminary values m_0 and β_0 are determined.

* See *Biometrika*, Vol. XII, p. 240.

In finding these values M_1 should first be found with great accuracy, remembering the difficulties with regard to even the first moment of curves with a finite initial ordinate and an abrupt departure from that ordinate. M_2 must be found with the same accuracy.

Now n_1 being the observed number in the first group and μ_1', μ_2' its first and second moment coefficients about $t=0$; μ_1'', μ_2'' the first and second moment coefficients of the remainder, $(N-n_1)$ about $t=0$, we have

$$Nc_1 = n_1\mu_1' + (N-n_1)\,\mu_1'',$$

$$Nc_2 = n_1\mu_2' + (N-n_1)\,\mu_2'',$$

whence c_1 and c_2 will be determinable from the observations: for μ_1' and μ_2' are to be given the values they would have for β_0 and m_0, and $(N-n_1)\,\mu_1''$ and $(N-n_1)\,\mu_2''$ are M_1 and M_2 respectively.

Let $\bar{n}_1$ be the *theoretical* frequency in the range $t=\beta_0$ to γ, then

$$\bar{n}_1 = \int_{\beta_0}^{\gamma} \frac{N}{m_0}\, e^{-\frac{(t-\beta_0)}{m_0}}\, dt = N\left(1 - e^{-\frac{(\gamma-\beta_0)}{m_0}}\right).$$

$$\bar{n}_1\,\mu_1' = \int_{\beta_0}^{\gamma} \frac{N}{m_0}\, e^{-\frac{(t-\beta_0)}{m_0}}\, t\, dt = N\left\{\beta_0 + m_0 - (\gamma + m_0)\, e^{-\frac{(\gamma-\beta_0)}{m_0}}\right\},$$

$$\bar{n}_1\,\mu_2' = \int_{\beta_0}^{\gamma} \frac{N}{m_0}\, e^{-\frac{(t-\beta_0)}{m_0}}\, t^2 dt = N\left\{(\beta_0+m_0)^2 + m_0^2 - [(\gamma+m_0)^2 + m_0^2]\, e^{-\frac{(\gamma-\beta_0)}{m_0}}\right\}$$

Thus

$$\mu_1' = \frac{\beta_0 + m_0 - (\gamma + m_0)\, e^{-\frac{(\gamma-\beta_0)}{m_0}}}{1 - e^{-\frac{(\gamma-\beta_0)}{m_0}}} \qquad \text{.....(vi)}$$

$$\mu_2' = \frac{(\beta_0+m_0)^2 + m_0^2 - \{(\gamma+m_0)^2 + m_0^2\}\, e^{-\frac{(\gamma-\beta_0)}{m_0}}}{1 - e^{-\frac{(\gamma-\beta_0)}{m_0}}} \qquad \text{.....(vii).}$$

Now knowing c_1 and c_2 from the observations in this way, we have from the whole theoretical curve $c_1 = \beta + m$ and $c_2 = (\beta+m)^2 + m^2$, and accordingly

$$m^2 = c_2 - c_1^2 \text{ or } m = \sqrt{c_2 - c_1^2} \qquad \text{.....(viii),}$$

$$\beta = c_1 - m \text{ or } \beta = c_1 - \sqrt{c_2 - c_1^2} \qquad \text{.....(ix).}$$

These give the second approximations to β and m, and if necessary we can use these values to redetermine μ_1' and μ_2' and proceed to a third approximation.

We will now illustrate the process on the experimental data for the frequency of intervals in an indefinitely long time. The smallest interval recorded was ·39 sec., and it seemed that β was fluctuating slightly throughout the experiment. To make the influence of the first group as small as possible, γ was taken to be ·5 so that the first group would be from β to ·5 and succeeding ones, ·5 to 1·5, 1·5 to 2·5 and so on. To find β and m, the first and second moment coefficients of

the distribution excluding the first group are found about ·5, using abruptness coefficients. These are transferred to $t = 0$, giving

$$\mu_1'' = 2{\cdot}929{,}7114, \qquad \mu_2'' = 14{\cdot}781{,}9089,$$

which give M_1 and M_2 respectively. Using these values the first approximations to β and m (i.e. β_0 and m_0) are, by equations (iv) and (v),

$$m_0 = 2{\cdot}429{,}7114, \qquad \beta_0 = {\cdot}428{,}8488.$$

Then the first and second moment coefficients of the first group about $t = 0$ (μ_1' and μ_2') are given the values they would have for β_0 and m_0, and these are, by (vi) and (vii),

$$\mu_1' = {\cdot}464{,}2481, \qquad \mu_2' = {\cdot}215{,}9042,$$

leading to $$c_1 = 2{\cdot}858{,}5599, \qquad c_2 = 14{\cdot}361{,}5450,$$

and finally by (viii) and (ix) the second approximations to β and m.

If it be thought necessary, these values can be used to redetermine μ_1' and μ_2' and hence third approximations to β and m. The results were in our case:

	β	m
2nd approximations	·370,552,	2·488,007,
3rd approximations	·368,867,	2·488,841.

The second approximations would be quite sufficiently accurate for most purposes.

The areas of the curve $y = \dfrac{N}{m} e^{-\frac{(t-\beta)}{m}}$ corresponding to the observed frequencies can be easily found when m and β are known.

The usual χ^2 test of goodness of fit* gave a very low value for P and it was clear that this was largely due to the very bad agreement of the first two groups (Table I). Clubbing these together (i.e. taking the first group from β to 1·5) gave the much higher value $P = {\cdot}222$. It seemed possible that this disagreement at the beginning of the range might be due to the fact that too few observations had been included in the first group and to test this the material was regrouped on the bases β to 1, 1 to 2 etc. This arrangement gave the constants

	β	m
2nd approximations	·362,918,	2·495,527,

the intermediate steps being

$$\begin{aligned} \mu_1'' &= 3{\cdot}390{,}0644, & \mu_2'' &= 17{\cdot}848{,}5430, \\ m_0 &= 2{\cdot}390{,}0644, & \beta_0 &= {\cdot}468{,}3801, \\ \mu_1' &= {\cdot}724{,}3450, & \mu_2' &= {\cdot}548{,}1725, \\ c_1 &= 2{\cdot}858{,}4446, & c_2 &= 14{\cdot}398{,}3602. \end{aligned}$$

Even without a knowledge of the probable errors of β and m we can safely say that the differences between these values and those found for the other grouping

* *Tables for Statisticians and Biometricians*, Table XII.

TABLE I.

Lengths of Intervals in an indefinitely long time, starting with a $\beta - \cdot 5$ grouping.

Frequency	Lengths of intervals in seconds										
	β—·5	·5—1·5	1·5—	2·5—	3·5—	4·5—	5·5—	6·5—	7·5—	8·5—	9·5—
Observed	106	1,217	845	475·5	353·5	221·5	143·5	95·5	65·5	59	29·5
Calculated	186·2	1,154·0	772·1	516·5	345·6	231·2	154·7	103·5	69·2	46·3	31·0

	Lengths of intervals in seconds										
10·5—	11·5—	12·5—	13·5—	14·5—	15·5—	16·5—	17·5—	18·5—	19·5—	Over 20·5	Totals
17·5	11	7	8	8	5	1	2	·5	·5	1	3,673
20·7	13·9	9·3	6·2	4·2	2·8	1·9	1·2	·8	·6	1·1	3,673

are not significant. The curve is shown fitted to the observed frequencies in the case when the first group is from β to 1 (Fig. I) and the areas are compared in Fig. II. Again there is a marked disagreement between the theoretical and observed frequencies of the first two groups; combining these, as before, gives for goodness of fit $P = \cdot 193$, so that regrouping the material has not given a better fit. The curve does not fit the observations well, but there can be little doubt that this is due to the conditions of the experiment and not to the form of the curve or the methods of finding β and m. β may have fluctuated considerably throughout the experiment and this would account for the bad fit at the beginning of the range; the longer intervals not being affected.

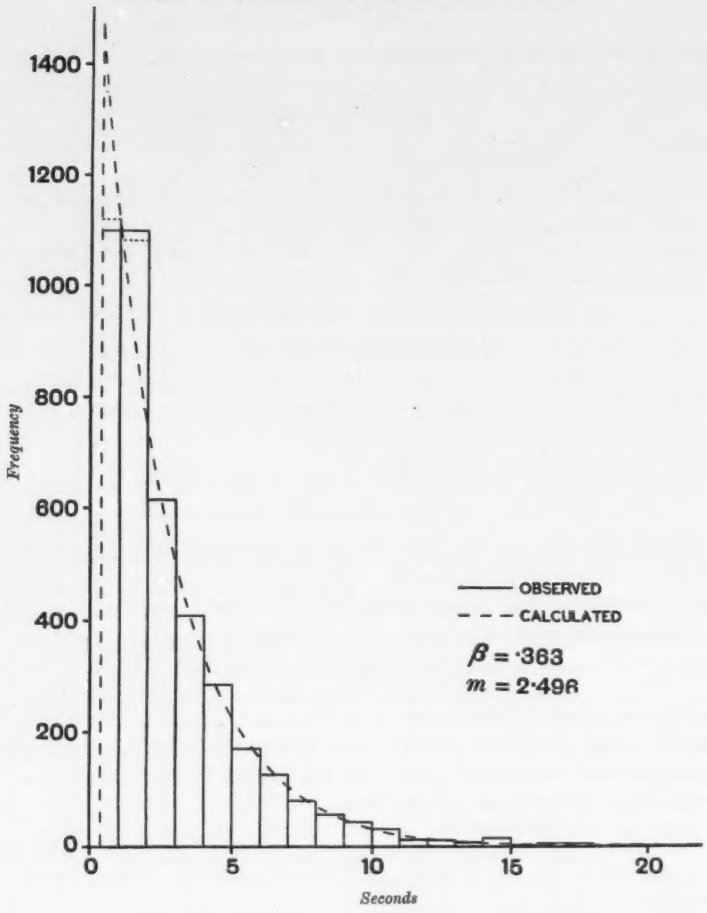
By finding the constants β and m from a sample only of the whole material we may get some idea of the way in which they were fluctuating. Taking two of the six tapes gave

β m

·354,020, 2·587,989.

These values differ appreciably from those found from all the data, but as we have at present no knowledge of the probable errors of β and m, it is not possible to say how significant these differences are. It would be more satisfactory if the above law of distribution of random intervals could be justified empirically with material which was more homogeneous and for which the "closed-time" β was more nearly constant.

Fig. I. Lengths of intervals in an indefinitely long time.



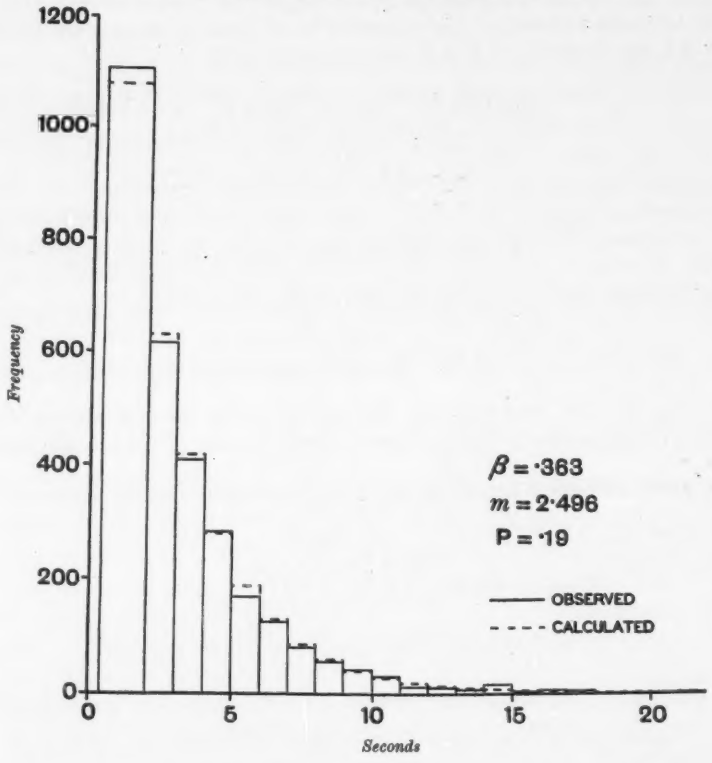
III. *Frequency of Occurrences in limited Periods of Time.*

The second question to be considered is that of the frequency of occurrences of the event in a limited period T. From a practical point of view this is, perhaps, a more important one than the preceding.

(*a*) It will be shown that the hypothesis has to be made that there was no occurrence within the "closed-time" β before the beginning of the period T, so that the material used is only a portion of that which can be used when considering events occurring in an indefinitely long time. Now, in the mathematical theory

m and β are rigidly constant quantities, but in actual practice they fluctuate slightly during the experiments or observations, and also the material is probably subject to fairly large errors of random sampling. Thus it by no means follows that the values of β and m found from the whole of the data will agree very closely with those obtained in another phase of the investigations when only a portion of the data can be used. It is therefore very essential to obtain m and β from each phase of the work independently, and to test the accordance of the values thus obtained. If the material given refer only to a limited period, the moment method of finding β and m, from the frequency of intervals in an indefinitely long time, cannot be used.

Fig. II. Lengths of intervals in an indefinitely long time.



We have to investigate the probability of n occurrences in a limited time T on the hypothesis that there was not an occurrence within the "closed-time" β before the beginning of the period T. Let the times of occurrence be $t_1, t_2 \ldots t_n$ from the beginning of the period and let $dt_1, dt_2 \ldots dt_n$ be indefinitely small intervals round

$t_1, t_2 \ldots t_n$. Let $f(\beta)$ be the chance of no occurrence happening during the interval β before the start of the period T. Then the chance of the observed event is

$$\delta C = f(\beta)\, e^{-\frac{t_1}{m}} \frac{dt_1}{m} e^{-\frac{(t_2-t_1-\beta)}{m}} \frac{dt_2}{m} e^{-\frac{(t_3-t_2-\beta)}{m}} \frac{dt_3}{m} \ldots \frac{dt_n}{m}\, q(t_n),$$

where $q(t_n)$ is the chance of no event occurring between t_n and T. If $t_n + \beta < T$, this is $e^{-\frac{(T-t_n-\beta)}{m}}$, but if $t_n + \beta \geqslant T$, this is certainty, for the time between t_n and T will be closed. Accordingly there is discontinuity in the form of the function $q(t_n)$ and we leave it for the moment as $q(t_n)$. We have therefore for the chance

$$\delta C = f(\beta) \frac{e^{+\frac{(n-1)\beta}{m}}}{m^n}\, dt_1 dt_2 \ldots dt_n\, q(t_n)\, e^{-\frac{t_n}{m}}.$$

We have now to integrate this expression for all possible values of the t's. The limits of t_n are clearly $t_{n-1} + \beta$ to T and our first integral

$$\begin{aligned}
&= \int_{t_{n-1}+\beta}^{T} dt_n\, q(t_n) = \int_{t_{n-1}+\beta}^{T-\beta} q(t_n)\, e^{-\frac{t_n}{m}} dt_n + \int_{T-\beta}^{T} e^{-\frac{t_n}{m}} \times 1 \times dt_n \\
&= \int_{t_{n-1}+\beta}^{T-\beta} e^{-\frac{(T-\beta)}{m}}\, dt_n + m\left(e^{-\frac{(T-\beta)}{m}} - e^{-\frac{T}{m}}\right) \\
&= e^{-\frac{(T-\beta)}{m}} \left\{(T - t_{n-1} - 2\beta) + m\left(1 - e^{-\frac{\beta}{m}}\right)\right\}.
\end{aligned}$$

Having thus freed ourselves from t_n our chance now becomes

$$\delta C' = f(\beta) \frac{e^{-\frac{(T-n\beta)}{m}}}{m^n}\, dt_1 dt_2 \ldots dt_{n-1} \left\{(T - t_{n-1} - 2\beta) + m\left(1 - e^{-\frac{\beta}{m}}\right)\right\}.$$

Now t_{n-1} must be integrated from $t_{n-2} + \beta$ up to β of the final position of t_n; this was $(T - \beta)$ for the first part of the integral of t_n, i.e. $(T - t_{n-1} - 2\beta)$, and T for the second part, i.e. $m\left(1 - e^{\frac{\beta}{m}}\right)$.

Thus our δt_{n-1} integral is

$$\begin{aligned}
&\int_{t_{n-2}+\beta}^{T-2\beta} (T - t_{n-1} - 2\beta)\, dt_{n-1} + m\left(1 - e^{-\frac{\beta}{m}}\right) \int_{t_{n-2}+\beta}^{T-\beta} dt_{n-1} \\
&\qquad = \frac{1}{2!}(T - t_{n-2} - 3\beta)^2 + m\left(1 - e^{-\frac{\beta}{m}}\right)(T - t_{n-2} - 2\beta).
\end{aligned}$$

Continuing this process we reach finally

$$C = f(\beta) \left\{ e^{-\frac{(T-n\beta)}{m}} \frac{1}{n!} \left(\frac{T-n\beta}{m}\right)^n + \left(1 - e^{-\frac{\beta}{m}}\right) e^{-\frac{(T-n\beta)}{m}} \frac{1}{(n-1)!} \left(\frac{T-(n-1)\beta}{m}\right)^{n-1} \right\} \qquad \ldots\ldots(x).$$

Now it is clear that this is a very complicated expression for the frequency of n occurrences and will be hard to deal with if m and β are as a rule unknown and

have to be found from the observations. The expression will be still more complicated if we introduce the frequency of n intervals in T, when an event has occurred in the interval β before T starts. For this case we shall have to replace $f(\beta)$ by $1-f(\beta)$, multiply by $\frac{dT}{m}$ and integrate from $T-\beta$ to T, or this chance will be

$$C'' = \{1-f(\beta)\}\left\{\int_{T-\beta}^{T}\left[\frac{e^{-\frac{(T-n\beta)}{m}}}{n!}\left(\frac{T-n\beta}{m}\right)^n + \left(1-e^{-\frac{\beta}{m}}\right)\frac{e^{-\frac{(T-n\beta)}{m}}}{(n-1)!}\left(\frac{T-(n-1)\beta}{m}\right)^{n-1}\right]dt\right\}$$

$$= \{1-f(\beta)\}\left\{\int_{\frac{T-(n+1)\beta}{m}}^{\frac{T-n\beta}{m}} \frac{e^{-x}x^n}{n!}\,dx + \int_{\frac{T-n\beta}{m}}^{\frac{T-(n-1)\beta}{m}} \left(e^{\frac{\beta}{m}}-1\right)\frac{e^{-x}x^{n-1}}{(n-1)!}\,dx\right\} \quad \text{.....(xi).}$$

These integrals could be integrated out by parts, but the process would be lengthy.

Now in a long time the mean interval between events is $\beta+m$, and the mean occupied space is β. Hence the chance that we start our epoch at an occupied space is $\frac{\beta}{\beta+m}$ and at an unoccupied space is $\frac{m}{(\beta+m)} = f(\beta)$.

(*b*) The mathematical theory is thus complete but is clearly of too complicated a character for much serviceable application. To test its accuracy experimentally it is desirable to throw out from our observations all sets of T in which an event occurs within β before the start of T or within β before the end of T. If β be relatively small both $1-f(\beta)$ and $1-e^{-\frac{\beta}{m}}$ will be small and this will not involve the rejection of a large number of T-periods.

When this is done, our formula reduces to

$$C = \frac{m}{\beta+m}\,\frac{e^{-\frac{(T-n\beta)}{m}}}{n!}\left(\frac{T-n\beta}{m}\right)^n.$$

The curve of frequency of periods T with n occurrences will accordingly be of the form

$$f_n = \frac{N\dfrac{e^{-\frac{(T-n\beta)}{m}}}{n!}\left(\dfrac{T-n\beta}{m}\right)^n}{\sum_0^{n_0}\dfrac{e^{-\frac{(T-n\beta)}{m}}}{n!}\left(\dfrac{T-n\beta}{m}\right)^n} \quad \text{.....(xii),}$$

where N = total number of periods of length T and n_0 = largest integer in $\frac{T}{\beta}$.

To simplify this further we ought to find a finite value for the series

$$\sum_0^{n_0}\frac{(a-nb)^n}{n!}, \text{ where } n_0 = \text{integer part of } \frac{a}{b}.$$

There appears to be no discussion of such a series, nor any likelihood of its being expressible in a simple form.

To find the β and m we proceed as follows:

$$f_0 = Ne^{-\frac{T}{m}}, \text{ and write } a = \frac{T}{m}e^{\frac{\beta}{m}},\ b = \frac{\beta}{m}e^{\frac{\beta}{m}}.$$

Accordingly
$$\frac{f_n}{f_0} = \frac{(a-nb)^n}{n!},$$

and
$$\chi_n = \left(\frac{n!\,f_n}{f_0}\right)^{\frac{1}{n}} = a - nb.$$

The weight w_n of χ_n will be inversely as the square of its standard deviation. But

$$\frac{\delta\chi_n}{\chi_n} = \frac{1}{n}\left(\frac{\delta f_n}{f_n} - \frac{\delta f_0}{f_0}\right),$$

and f_n and f_0 being frequency groups,

$$\sigma^2_{f_n} = f_n\left(1 - \frac{f_n}{N}\right), \quad \sigma^2_{f_0} = f_0\left(1 - \frac{f_0}{N}\right),$$

$$\text{Mean } (\delta f_n \delta f_0) = -\frac{f_n f_0}{N}.$$

Thus approximately

$$\frac{\sigma^2_{\chi_n}}{\chi_n^2} = \frac{1}{n^2}\left\{\frac{1}{f_n}\left(1 - \frac{f_n}{N}\right) + \frac{1}{f_0}\left(1 - \frac{f_0}{N}\right) + \frac{2f_nf_0}{N}\cdot\frac{1}{f_nf_0}\right\}$$

$$= \frac{1}{n^2}\left(\frac{1}{f_n} + \frac{1}{f_0}\right) \quad \text{.....(xiii).}$$

Accordingly we may take

$$w_n = \frac{n^2}{\chi_n^2}\frac{f_0 f_n}{(f_0 + f_n)} \quad \text{.....(xiv).}$$

To find the best values of a and b we have thus to make

$$u^2 = \sum_0^{n_0}(a - nb - \chi_n)^2 w_n \text{ a minimum,}$$

or we reach

$$\left.\begin{aligned} a\sum_0^{n_0}(w_n) - b\sum_0^{n_0}(nw_n) &= \Sigma(\chi_n w_n) \\ a\sum_0^{n_0}(nw_n) - b\sum_0^{n_0}(n^2 w_n) &= \Sigma(n\chi_n w_n) \end{aligned}\right\} \text{.....(xv).}$$

These equations have not been found too laborious in practice. When a and b are determined, we have

$$\beta = \frac{b}{a}T \text{.....(xvi)}$$

and then if $z = \frac{\beta}{m}$, we must find z and eventually m from

$$b = ze^z \text{ .....(xvii).}$$

(*c*) The above method of finding β and m cannot be used when the observed frequency f_0 is zero, and we should not expect it to give good results if f_0 were small. A more general method is required to cover these cases.

We have, by (xii),

$$f_n = C\,\frac{e^{-\frac{(T-n\beta)}{m}}}{n!}\left(\frac{T-n\beta}{m}\right)^n,$$

where C is a constant.

Accordingly

$$\frac{f_n^2}{f_{n-1}f_{n+1}} = \frac{n+1}{n}\,\frac{(T-n\beta)^{2n}}{(T-(n-1)\beta)^{n-1}(T-(n+1)\beta)^{n+1}},$$

or

$$\chi_n' = \log_e\left(\frac{f_n^2 n}{f_{n-1}f_{n+1}(n+1)}\right)$$

$$= 2n\log_e(T-n\beta) - (n-1)\log_e(T-(n-1)\beta) - (n+1)\log_e(T-(n+1)\beta).$$

Taking the weight w_n' of χ_n' to be inversely as the square of its standard deviation,

$$\delta\chi_n' = \frac{2\delta f_n}{f_n} - \frac{\delta f_{n+1}}{f_{n+1}} - \frac{\delta f_{n-1}}{f_{n-1}}.$$

Then

$$\sigma^2_{\chi_n} = \frac{4}{f_n^2}\sigma^2_{f_n} + \frac{\sigma^2_{f_{n+1}}}{f^2_{n+1}} + \frac{\sigma^2_{f_{n-1}}}{f^2_{n-1}} - \frac{4[\delta f_n \delta f_{n+1}]}{f_n f_{n+1}} - \frac{4[\delta f_n \delta f_{n-1}]}{f_n f_{n-1}} + \frac{2[\delta f_{n+1}\delta f_{n-1}]}{f_{n+1}f_{n-1}},$$

the square brackets denoting mean values.

Thus approximately

$$\sigma^2_{\chi_n'} = \frac{4}{f_n}\left(1-\frac{f_n}{N}\right) + \frac{1}{f_{n+1}}\left(1-\frac{f_{n+1}}{N}\right) + \frac{1}{f_{n-1}}\left(1-\frac{f_{n-1}}{N}\right) + \frac{4}{N} + \frac{4}{N} - \frac{2}{N}$$

$$= \frac{4}{f_n} + \frac{1}{f_{n+1}} + \frac{1}{f_{n-1}},$$

or

$$w_n' = \frac{f_n f_{n+1} f_{n-1}}{f_n f_{n-1} + f_n f_{n+1} + 4 f_{n-1} f_{n+1}} \qquad \text{.....(xviii).}$$

To find the best value of β we have to make

$$u^2 = \sum_1^{n_2} w_n' \{2n\log_e(T-n\beta) - (n-1)\log_e(T-(n-1)\beta) - (n+1)\log_e(T-(n+1)\beta) - \chi_n'\}^2$$

a minimum, giving

$$0 = \sum_1^{n_2}\left\{w_n'[2\log_e(T-n\beta) - (n-1)\log_e(T-(n-1)\beta) - (n+1)\log_e(T-(n+1)\beta) - \chi_n'] \times \left(\frac{2n^2}{T-n\beta} - \frac{(n-1)^2}{T-(n-1)\beta} - \frac{(n+1)^2}{T-(n+1)\beta}\right)\right\} \qquad \text{.....(xix).}$$

β can be found from this equation by making successive approximations, but the process is a lengthy one. Logarithms can be taken to the base 10 since this change will not affect the relative weights.

We have next to find m when β is known.

We have

$$\psi_n = \log_{10}\frac{f_n}{f_{n-1}} = \frac{\beta}{m}\log_{10}e - \log_{10}n - \log_{10}m + n\log_{10}(T-n\beta) - (n-1)\log_{10}(T-(n-1)\beta).$$

Proceeding as before, the weight w_n'' of ψ_n will be $\frac{f_n f_{n-1}}{f_n + f_{n-1}}$ and for the best value of m,

$$u^2 = \sum_1^{n_0} w_n'' \left\{\frac{\beta}{m}\log_{10}e - \log_{10}n - \log_{10}m + n\log_{10}(T-n\beta) - (n-1)\log_{10}(T-(n-1)\beta) - \psi_n\right\}^2$$

has to be made a minimum. Thus we reach

$$0 = \sum_1^{n_0}\left\{w_n''\left(\frac{\beta}{m}\log_{10}e - \log_{10}n - \log_{10}m + n\log_{10}(T-n\beta) - (n-1)\log_{10}(T-(n-1)\beta) - \psi_n\right)\times\left(\frac{\beta\log_{10}e}{m} + \frac{1}{\log_e 10\,.\,m}\right)\right\} \quad \text{.....(xx)},$$

from which m can be found since β is known.

(*d*) We have not succeeded in finding the mean number of occurrences or the standard deviation of occurrences, when they are limited terminally as above described.

Corollary. If we take $\beta = 0, f(\beta) = 1$, then our formula reduces to

$$f_n = N\frac{e^{-\frac{T}{m}}}{n!}\left(\frac{T}{m}\right)^n \quad \text{.....(xxi)},$$

the well-known Poisson expansion, which gives the frequency of n occurrences in time T.

(*e*) We come now to the application of the above to the experimental data. The tapes were divided into 10-second periods and all those were rejected which were either preceded by an occurrence in time less than β, or in which there was an occurrence within β of the end of the period. But at this stage the value of β which the sample being dealt with would give was, of course, not known. That found from the distribution of lengths of intervals in an indefinitely long time was used, since the difference between the two would certainly be small, but usually in practice such a close approximation would not be available.

The constants β and m have first to be found. Of a total of 931 periods there were 22 in which there was no occurrence of the event, so the $\chi_n = \left(\frac{n!f_n}{f_0}\right)^{\frac{1}{n}}$ method could be safely used. The values of χ_n calculated for all possible values of n and weighted by (xiv) lead to the two equations (xv) and hence

$$a = 4{\cdot}444{,}441, \quad b = {\cdot}157{,}272.$$

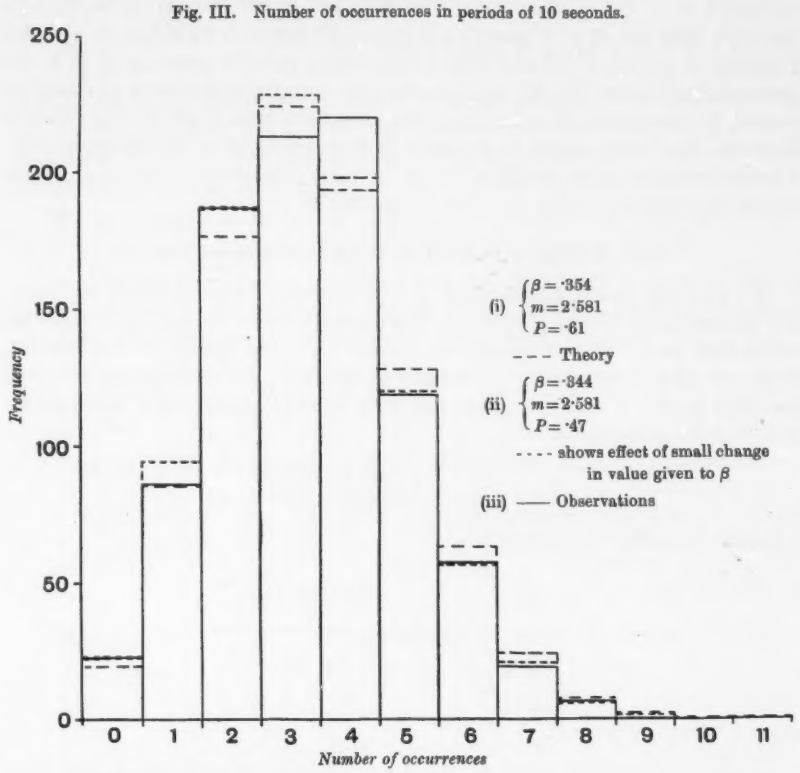
These by (xvi) give $\beta = \cdot 353{,}862$ and equation (xvii) is

$$\bar{1}\cdot 196{,}6514 = \log z + (z \times \cdot 434{,}2945),$$

where $z = \dfrac{\beta}{m}$, and hence $m = 2\cdot 580{,}641$.

These values of β and m differ appreciably from those found from the whole material; this possibly being due to the fact that the selected portion was not a random sample. The frequency of periods T containing n occurrences is given by (xii) and Fig. III shows that the theoretical fits the observed distribution quite satisfactorily. The P for goodness of fit is $\cdot 614$.

Fig. III. Number of occurrences in periods of 10 seconds.



The alternative $\chi_n' = \log\left(\dfrac{f_n^2}{f_{n-1}f_{n+1}}\dfrac{n}{n+1}\right)$ method gave

$$\beta = \cdot 393{,}772, \quad m = 2\cdot 495{,}831,$$

and the following distribution.

For goodness of fit this table gives $P = \cdot 80$ which is a considerable improvement on the previous value found ($P = \cdot 61$), but the first method is very much less

TABLE II.

Frequency of Periods of 10 *seconds with n occurrences.*

Frequency	$n=0$	1	2	3	4	5	6	7	8	9 and over	Totals
Observed	22·0	86·0	187·0	214·0	221·0	119·0	57·0	19·0	6·0	0·0	931·0
Calculated	19·5	87·7	189·3	229·6	197·8	123·1	57·2	20·2	5·5	1·4	931·3

laborious than the second and it will generally be quite accurate enough. Using the values of β and m found from the frequency of intervals in an indefinitely long time with the β to ·5 grouping (i.e. $\beta = ·370{,}552$, $m = 2·488{,}007$) to give the frequency of periods of 10 secs. with n occurrences, gave for goodness of fit to the observed distribution $P = ·67$, which shows that it may not be always necessary in practice to recalculate the constants when they have been found for one phase of the work. But it will always be necessary to do so unless, as in the case considered, β is small relative to the length of the period taken, and there are thus few periods having an occurrence within β of their beginning or end.

IV. *Lengths of Intervals in limited Periods of Time.*

We now pass to the consideration of the number of intervals which will occur in a limited period of time T, and also to the distribution of the lengths of intervals within that period when there are, say, n intervals in the period. In this case one occurrence and no occurrences will not give an interval. If n' = number of intervals, then they involve $n'+1$ occurrences, and if $F_{n'}$ be the frequency of n' intervals, we shall have $F_{n'}$ proportional to

$$\frac{e^{-\frac{(T-(n'+1)\beta)}{m}}}{(n'+1)!}\left(\frac{T-(n'+1)\beta}{m}\right)^{n'+1}$$

or for the frequency of intervals N',

$$F''_{n'} = \frac{N'\dfrac{e^{-\frac{(T-(n'+1)\beta)}{m}}}{(n'+1)!}\left(\dfrac{T-(n'+1)\beta}{m}\right)^{n'+1}}{\sum_{1}^{n_0'}\dfrac{e^{-\frac{(T-(n'+1)\beta)}{m}}}{(n'+1)!}\left(\dfrac{T-(n'+1)\beta}{m}\right)^{n'+1}} \quad \text{.....(xxii),}$$

where n_0' = integer part of $\dfrac{T}{\beta}-1$.

If N'' = total frequency when we count the periods T in which there are *no* intervals, then since no intervals may arise from 0 or 1 occurrence, we shall have

$$F_0' = \frac{N''e^{-\frac{T}{m}}\left(1+e^{\frac{\beta}{m}}\dfrac{T-\beta}{m}\right)}{\sum_{-1}^{n_0'}\dfrac{e^{-\frac{(T-(n'+1)\beta)}{m}}}{(n'+1)!}\left(\dfrac{T-(n'+1)\beta}{m}\right)^{n'+1}} \quad \text{.....(xxiii),}$$

while

$$F''_{n'} = \frac{N'' e^{-\frac{(T-(n'+1)\beta)}{m}} \left(\frac{T-(n'+1)\beta}{m}\right)^{n'+1}}{(n'+1)!} \Bigg/ \sum_{-1}^{n_0'} \frac{e^{-\frac{(T-(n'+1)\beta)}{m}} \left(\frac{T-(n'+1)\beta}{m}\right)^{n'+1}}{(n'+1)!} \quad \text{.....(xxiii)}^{\text{bis}}.$$

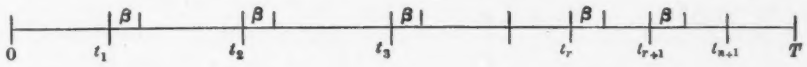
If we are only given F_0' as a whole, and the several $F''_{n'}$, it is difficult to see any ready solution of the problem. If we are given the part F_0'' of F_0' which covers no occurrences in T we can proceed exactly as in the previous problem, writing

$$\chi'_{n'} = \left\{\frac{F''_{n'}(n'+1)!}{F_0''}\right\}^{\frac{1}{n'+1}} = a-(n'+1)b,$$

and the solution proceeds just as before. If we are given F_0''', the number of cases of a single occurrence, but not F_0'', the equation seems as intractable as when we only know F_0' as a whole. Yet this seems a not impossible case in practice, namely one in which there would be no record unless at least one occurrence happened.

We shall now determine the frequency distribution for intervals of size t occurring in the period T which we will suppose to contain n' intervals.

We enquire first what is the chance that the rth interval of these n' intervals is of magnitude t. We shall again exclude the possibility of an occurrence having happened at an interval less than β before the start of T.



Then, as before, the probability of the system being as in the diagram above is

$$f(\beta)\, e^{-\frac{t_1}{m}} \frac{dt_1}{m} e^{-\frac{(t_2-t_1-\beta)}{m}} \frac{dt_2}{m} e^{-\frac{(t_3-t_2-\beta)}{m}} \frac{dt_3}{m} \dots \frac{dt_r}{m} e^{-\frac{(t_{r+1}-t_r-\beta)}{m}} \frac{dt_{r+1}}{m} \dots$$
$$\dots e^{-\frac{(t_{n+1}-t_n-\beta)}{m}} \frac{dt_{n+1}}{m}\, q(t_{n+1}).$$

Here $t_{r+1}-t_r = t$ and is to remain constant while we integrate for all the $t_1 \dots t_{n+1}$ but t_{r+1}. Clearly the above probability can be written

$$f(\beta)\frac{1}{m^{n+1}} e^{-\frac{(T-n\beta)}{m}} dt_1 dt_2 \dots dt_{n+1} e^{-\frac{(t_{n+1}-T)}{m}} q(t_{n+1}).$$

We first integrate t_{n+1} from $t_{n+\beta}$ to T, i.e. we determine

$$\int_{t_n+\beta}^{T} e^{-\frac{(t_{n+1}-T)}{m}} q(t_{n+1})\, dt_{n+1}$$
$$= \int_{t_n+\beta}^{T-\beta} e^{-\frac{(t_{n+1}-T)}{m}} e^{-\frac{(T-t_{n+1}-\beta)}{m}} dt_{n+1} + \int_{T-\beta}^{T} e^{-\frac{(t_{n+1}-T)}{m}} dt_{n+1}$$
$$= e^{\frac{\beta}{m}}(T-t_n-2\beta) + m\left(e^{\frac{\beta}{m}}-1\right).$$

Thus as the result of our first integration we have

$$f(\beta)\frac{e^{-\frac{(T-(n+1)\beta)}{m}}}{m^{n+1}}\,dt_1dt_2\ldots dt_n\left\{(T-t_n-2\beta)+m\left(1-e^{-\frac{\beta}{m}}\right)\right\}.$$

We have next to integrate t_n from $t_{n-1}+\beta$ up to $t_{n+1}-\beta$, where t_{n+1} is given in the first part of the subject of integration the value $T-\beta$ and in the second T. Accordingly the integration leads to

$$f(\beta)\frac{e^{-\frac{(T-(n+1)\beta)}{m}}}{m^{n+1}}\,dt_1dt_2\ldots dt_{n-1}\left\{\frac{(T-t_{n-1}-3\beta)^2}{2!}+m\left(1-e^{-\frac{\beta}{m}}\right)(T-t_{n-1}-2\beta)\right\}.$$

If we continue this process up to dt_{r+1} we have

$$f(\beta)\frac{e^{-\frac{(T-(n+1)\beta)}{m}}}{m^{n+1}}\,dt_1dt_2\ldots dt_rdt_{r+1}\left\{\frac{[T-t_{r+1}-(n-r+1)\beta]^{n-r}}{(n-r)!}\right.$$
$$\left.+m\left(1-e^{-\frac{\beta}{m}}\right)\frac{[T-t_{r+1}-(n-r)\beta]^{n-r-1}}{(n-r-1)!}\right\}.$$

Now we do not integrate with regard to dt_{r+1} but put $t_{r+1}=t+t_r$ and proceed to integrate with regard to t_r from $t_r=t_{r-1}+\beta$ to $T-t-(n+r+1)\beta$ in the first subject of integration and to $T-t-(n-r)\beta$ in the second subject. We have accordingly

$$f(\beta)\frac{e^{-\frac{(T-(n+1)\beta)}{m}}}{m^{n+1}}\,dt_1dt_2\ldots dt_{r-1}\left\{\frac{[T-t_{r-1}-t-(n-r+2)\beta]^{n-r+1}}{(n-r+1)!}\right.$$
$$\left.+m\left(1-e^{-\frac{\beta}{m}}\right)\frac{[T-t_{r-1}-t-(n-r+1)\beta]^{n-r}}{(n-r)!}\right\}dt_{r+1}.$$

Continuing this process we have finally after writing dt for dt_{r+1}

$$C_t=f(\beta)\frac{e^{-\frac{(T-n(+1)\beta)}{m}}}{m^{n+1}}\left\{\frac{(T-t-n\beta)^n}{n!}+m\left(1-e^{-\frac{\beta}{m}}\right)\frac{(T-t-(n-1)\beta)^{n-1}}{(n-1)!}\right\}dt$$
.....(xxiv)

for the chance of an interval between t and $t+dt$. Now if we integrate this from $t=\beta$ to $t=T-n\beta$ for the first subject and to $t=T-(n-1)\beta$ for the second subject we have

$$\int C_t\,dt=f(\beta)\frac{e^{-\frac{(T-(n+1)\beta)}{m}}}{m^{n+1}}\left\{\frac{(T-(n+1)\beta)^{n+1}}{(n+1)!}+m\left(1-e^{-\frac{\beta}{m}}\right)\frac{(T-n\beta)^n}{n!}\right\},$$

which agrees with (x), if we remember that n intervals signify $n+1$ occurrences. In precisely the same way we get a form like (xi) (with n changed to $n+1$) by integrating with regard to T from $T-\beta$ to T and replacing $f(\beta)$ by $1-f(\beta)$. The result expresses the chance of any interval t occurring in the series of intervals when the period starts somewhere in a closed interval β. The great complexity of the resulting formula compels us again to simplify matters if β and

m are not *a priori* known by selecting those intervals only in which no event occurs in the period β before T starts or in the period β before the end of T.

In this case the chance C_t of a definite interval in an n interval period being between t and $t+dt$ is

$$C_t = f(\beta)\, e^{-\frac{(T-(n+1)\beta)}{m}} \frac{(T-t-n\beta)^n}{m^{n+1}\, n\,!}\, dt \qquad \text{.....(xxv).}$$

This chance is clearly the same whether the given interval occurs between the rth and $r+1$th events or between any other pair of events. It therefore represents the chance of an interval of the required length being in any position. But if we are going to tabulate the frequency of intervals there will be n cases in which such a length of interval could occur, and we shall have for our frequency surface

$$f_{n,t} \text{ varies as } e^{-\frac{(T-(n+1)\beta)}{m}} \frac{(T-t-n\beta)^n}{m^{n+1}(n-1)\,!}.$$

To obtain the constant of variation, C, we must find

$$N' = \sum_1^{n_0} \int_\beta^{T-n\beta} f_{n,t}\, dt = \sum_1^{n_0} \left\{ C \times e^{-\frac{(T-(n+1)\beta)}{m}} \frac{(T-(n+1)\beta)^{n+1}}{m^{n+1}(n+1)\,!}\, n \right\},$$

or,

$$C = \frac{N'}{\sum_1^{n_0} \left\{ e^{-\frac{(T-(n+1)\beta)}{m}} \frac{(T-(n+1)\beta)^{n+1}}{m^{n+1}(n+1)\,!}\, n \right\}},$$

where N' is the total number of *intervals*.

Thus finally

$$f_{n,t} = \frac{N' e^{-\frac{(T-(n+1)\beta}{m}} \frac{(T-t-n\beta)^n}{m^{n+1}(n-1)\,!}}{\sum_1^{n_0} \left\{ e^{-\frac{(T-(n+1)\beta]}{m}} \frac{(T-(n+1)\beta)^{n+1}}{m^{n+1}(n+1)\,!}\, n \right\}} \qquad \text{.....(xxv)}^{\text{bis}}$$

We proceed now to deduce some consequences from this result.

(i) If $\beta = 0$, $f(\beta) = 1$, and we no longer have difficulties about terminal conditions, and the chance that a given interval shall be of length between t and $t+dt$ in an n-interval period is

$$C_t = e^{-\frac{T}{m}} \frac{(T-t)^n}{m^{n+1}\, n\,!}\, dt \qquad \text{.....(xxvi).}$$

Integrating t from 0 to T

$$\int_0^T C_t dt = e^{-\frac{T}{m}} \frac{T^{n+1}}{m^{n+1}(n+1)\,!}.$$

This is the total chance of n intervals regardless of their size, and summing from $n=1$ to ∞,

$$\sum_1^\infty \int_0^T C_t dt = e^{-\frac{T}{m}} \sum_1^\infty \left\{ \frac{T^{n+1}}{m^{n+1}(n+1)\,!} \right\} = e^{-\frac{T}{m}} \left(e^{\frac{T}{m}} - 1 - \frac{T}{m} \right)$$

$$= 1 - e^{-\frac{T}{m}} - \frac{T}{m} e^{-\frac{T}{m}}$$

$$= \text{certainty} - \text{chance of } no \text{ event} - \text{chance of one event}$$

which it clearly should be.

(ii) Let us return to our fundamental equation (xxv)^{bis}, and let us first find the total frequency in an array corresponding to a given n. This will equal (if λ represent a constant)

$$N_n = \int_{\beta}^{T-n\beta} f_{n,t}\, dt = \lambda e^{-\frac{(T-(n+1)\beta)}{m}} \frac{(T-(n+1)\beta)^{n+1}}{m^{n+1}(n+1)!} n.$$

Next, let $\bar{t}_n$ be the mean interval for the array. This will be given by

$$N_n \bar{t}_n = \frac{\lambda e^{-\frac{(T-(n+1)\beta)}{m}}}{m^{n+1}(n-1)!} \int_{\beta}^{T-n\beta} t\,(T-t-n\beta)^n\, dt$$

$$= \frac{\lambda e^{-\frac{(T-(n+1)\beta)}{m}}}{m^{n+1}(n-1)!} \left\{ \frac{\beta\,(T-(n+1)\beta)^{n+1}}{n+1} + \frac{1}{(n+1)(n+2)} (T-(n+1)\beta)^{n+2} \right\}$$

$$= \frac{\lambda e^{-\frac{(T-(n+1)\beta)}{m}}}{m^{n+1}(n+1)!} \cdot \frac{n(T+\beta)}{n+2} (T-(n+1)\beta)^{n+1}.$$

Dividing out by the value first found for N_n we have

$$\bar{t}_n = \frac{T+\beta}{n+2} \qquad \text{.....(xxvii).}$$

The means of the arrays are therefore points on a rectangular hyperbola of which $t=0$ and $n=-2$ are the asymptotes, and that curve is one of the regression lines of the surface.

Again, if $\sigma_{t,n}$ be the standard deviation of the n array of intervals

$$\sigma^2_{t,n} + \bar{t}_n^{\,2} = \frac{(n+1)\int_{\beta}^{T-n\beta} t^2 (T-t-n\beta)^n}{(T-(n+1)\beta)^{n+1}}$$

$$= \beta^2 + \frac{2\beta}{n+2}(T-(n+1)\beta) + \frac{2(T-(n+1)\beta)^2}{(n+2)(n+3)},$$

whence, substituting for t_n, we find after some reductions

$$\sigma^2_{t,n} = \frac{(T-(n+1)\beta)^2 (n+1)}{(n+2)^2 (n+3)} \qquad \text{.....(xxviii).}$$

We see accordingly that the system is markedly heteroscedastic, the variability of the array decreasing rapidly with n.

For the special case of $\beta = 0$,

$$\bar{t}_n = \frac{T}{n+2}, \qquad \sigma_{t,n} = \frac{T}{n+2}\sqrt{\frac{n+1}{n+3}} \qquad \text{.....(xxix).}$$

Thus the mean and standard deviation of any array tend to equality.

We have further for the mean interval

$$\bar{t} = (T+\beta) \times \frac{\sum_1^{n_0} \left\{ e^{-\frac{(T-(n+1)\beta)}{m}} \frac{(T-(n+1)\beta)^{n+1}}{m^{n+1}(n+1)!} \frac{n}{n+2} \right\}}{\sum_1^{n_0} \left\{ e^{-\frac{(T-(n+1)\beta)}{m}} \frac{(T-(n+1)\beta)^{n+1}}{m^{n+1}(n+1)!} n \right\}} \qquad \text{.....(xxx).}$$

And doubtless σ_t could be found in a similar if more complicated form; but such expressions do not suggest any great hope of our being able to derive β and m from the mean and standard deviations of the intervals occurring in periods T.

In the same way the equations for the mean $\bar{n}_t$ and the standard deviation σ_t of the array of n's for values of t from t to $t+dt$ do not look very likely to lead anywhere in the case of a finite β, i.e.

$$\bar{n}_t = \frac{\overset{n_0}{\underset{1}{S}}\left\{e^{\frac{(n+1)\beta}{m}}\frac{(T-t-n\beta)^n}{m^{n+1}(n-1)!}n\right\}}{\overset{n_0}{\underset{1}{S}}\left\{e^{\frac{(n+1)\beta}{m}}\frac{(T-t-n\beta)^n}{m^{n+1}(n-1)!}\right\}} \quad \text{.....(xxxi),}$$

$$\sigma^2_{t,n} + \bar{n}_t^2 = \frac{\overset{n_0}{\underset{1}{S}}\left\{e^{\frac{(n+1)\beta}{m}}\frac{(T-t-n\beta)^n}{m^{n+1}(n-1)!}n^2\right\}}{\overset{n_0}{\underset{1}{S}}\left\{e^{\frac{(n+1)\beta}{m}}\frac{(T-t-n\beta)^n}{m^{n+1}(n-1)!}\right\}} \quad \text{.....(xxxii).}$$

(iii) All these results, however, admit of comparatively easy development when there is no "closed-time" β.

In this case for the frequency distribution of N intervals

$$f_{n,t} = \frac{\dfrac{N(T-t)^n}{m^{n+1}(n-1)!}}{\left\{1+e^{\frac{T}{m}}\left(\frac{T}{m}-1\right)\right\}} \quad \text{.....(xxxiii)}$$

$$= \frac{\lambda'(T-t)^n}{m^{n+1}(n-1)!}, \text{ say.}$$

We have already seen what $\bar{t}_n$ and $\sigma_{t,n}$ are. We will now find $\bar{n}_t$ and $\sigma_{n,t}$.

$$\overset{\infty}{\underset{1}{S}}\left\{\frac{\lambda'}{m}\frac{(T-t)^n}{m^n}\frac{dt}{(n-1)!}\right\} = \frac{(T-t)\lambda'}{m^2}\overset{\infty}{\underset{1}{S}}\left\{\frac{(T-t)^{n-1}}{(n-1)!}\frac{dt}{m^{n-1}}\right\}.$$

Or, $$N_t = \frac{\lambda'(T-t)}{m^2}\,dt\,e^{-\frac{T-t}{m}}*,$$

$$\bar{n}_t N_t = \lambda' \overset{\infty}{\underset{1}{S}}\left\{\frac{n(T-t)^n}{m^{n+1}(n-1)!}\right\}dt$$

$$= \lambda' dt\left[\frac{(T-t)^2}{m^3}\overset{\infty}{\underset{2}{S}}\left\{\frac{(T-t)^{n-2}}{(n-2)!\,m^{n-2}}\right\} + \frac{(T-t)}{m^2}\overset{\infty}{\underset{1}{S}}\left\{\frac{(T-t)^{n-1}}{(n-1)!\,m^{n-1}}\right\}\right]$$

$$= \lambda'\left[\frac{(T-t)^2}{m^3}e^{\frac{T-t}{m}} + \frac{(T-t)}{m^2}e^{\frac{T-t}{m}}\right]dt$$

* This shows us that the frequency curve for intervals t is

$$y = \frac{\dfrac{T-t}{m^2}e^{\frac{T-t}{m}}}{1+e^{\frac{T}{m}}\left(\frac{T}{m}-1\right)} \quad \text{.....(xxxiii)}^{bis}$$

If we make T infinite, this reduces to the well-known form $y = \frac{1}{m}e^{-\frac{t}{m}}$. We have determined its mean and standard deviation in the text.

Thus $$\bar{n}_t = \frac{T-t}{m} + 1 \quad \text{.....(xxxiv).}$$

There should be a direct proof of this since $\left(\frac{T-t}{m}\right)$ is clearly the mean number of occurrences in the interval $(T-t)$, or we ought to be able to show that the mean number of occurrences is one *less* than the mean number of intervals*.

The regression of n on t is thus seen to be linear, while that of t on n is hyperbolic.

Again,

$$N_t(\sigma^2_{n,t} + \bar{n}_t^2) = \lambda' \sum_1^\infty \left\{ \frac{n^2 (T-t)^n}{m^{n+1}(n-1)!} \right\} dt$$

$$= \lambda' \left\{ \frac{(T-t)^3}{m^4} \sum_3^\infty \left(\frac{(T-t)^{n-3}}{m^{n-3}(n-3)!} \right) + \frac{3(T-t)^2}{m^3} \sum_2^\infty \left(\frac{(T-t)^{n-2}}{m^{n-2}(n-2)!} \right) + \frac{(T-t)}{m^2} \sum_1^\infty \left(\frac{(T-t)^{n-1}}{m^{n-1}(n-1)!} \right) \right\} dt$$

$$= \lambda' \frac{e^{\frac{T-t}{m}}}{m} \left\{ \left(\frac{T-t}{m}\right)^3 + 3\left(\frac{T-t}{m}\right)^2 + \frac{T-t}{m} \right\} dt.$$

Accordingly, $$\sigma^2_{n,t} + \bar{n}_t^2 = \left(\frac{T-t}{m}\right)^2 + \frac{3(T-t)}{m} + 1,$$

and $$\sigma_{n,t} = \sqrt{\frac{T-t}{m}} \quad \text{.....(xxxv)}$$

The arrays are accordingly heteroscedastic.

Clearly, $$N\bar{n} = \lambda' \int_0^T \frac{(T-t)}{m^2} \left(\frac{T-t}{m} + 1 \right) e^{+\frac{T-t}{m}} dt,$$

whence integrating and substituting the value for λ' we find

$$\bar{n} = \frac{\frac{T^2}{m^2}}{\frac{T}{m} - 1 + e^{-\frac{T}{m}}} - 1 \quad \text{.....(xxxvi).}$$

To find σ_n we have

$$N(\sigma_n^2 + \bar{n}^2) = \sum_1^\infty \left\{ \lambda' \int_0^T \frac{(T-t)^n n^2}{m^{n+1}(n-1)!} dt \right\}$$

$$= \lambda' \sum_1^\infty \left\{ \frac{T^{n+1}}{m^{n+1}} \frac{n^2}{(n+1)(n-1)!} \right\} = \lambda' \sum_1^\infty \left\{ \frac{n^3}{(n+1)!} \frac{T^{n+1}}{m^{n+1}} \right\}$$

$$= \lambda' \left\{ \sum_2^\infty \left(\frac{T^3}{m^3} \frac{T^{n-2}}{m^{n-2}(n-2)} \right) + \sum_1^\infty \left(\frac{T}{m} \frac{T^n}{m^n n!} \right) - \sum_1^\infty \left(\frac{T^{n+1}}{m^{n+1}(n+1)!} \right) \right\}$$

$$= \lambda' \left\{ e^{\frac{T}{m}} \left(\frac{T^3}{m^3} + \frac{T}{m} - 1 \right) + 1 \right\}.$$

* The number of intervals is always one less than the number of occurrences, but then there are cases of no or one occurrence (giving no intervals) which affect the mean number of occurrences.

Thus
$$\sigma_n^2+\bar{n}^2=\frac{\left(\frac{T}{m}\right)^2+\frac{T}{m}-1+e^{-\frac{T}{m}}}{\frac{T}{m}-1+e^{-\frac{T}{m}}}$$
$$=1+\frac{\left(\frac{T}{m}\right)^2}{\frac{T}{m}-1+e^{-\frac{T}{m}}}$$

whence we deduce

$$\sigma_n=\frac{\frac{T}{m}}{\frac{T}{m}-1+e^{-\frac{T}{m}}}\sqrt{\frac{T}{m}\left(1+e^{-\frac{T}{m}}\right)-2\left(1-e^{-\frac{T}{m}}\right)} \quad \text{.....(xxxvii).}$$

We may now deduce $\bar{t}$,

$$N\bar{t}=\lambda' \overset{\infty}{S}_1\left\{\int_0^T\frac{(T-t)^n t}{m^{n+1}(n-1)!}dt\right\}$$
$$=\lambda'\left\{\overset{\infty}{S}_1\left(\frac{T^{n+2}n}{m^{n+1}(n+1)!}\right)-\overset{\infty}{S}_1\left(\frac{n(n+1)T^{n+2}}{m^{n+1}(n+2)!}\right)\right\}$$
$$=\lambda' m\left\{\frac{T^2}{m^2}\overset{\infty}{S}_1\left(\frac{T^n}{m^n n!}\right)-\frac{T}{m}\overset{\infty}{S}_1\left(\frac{T^{n+1}}{m^{n+1}(n+1)!}\right)-\frac{T^2}{m^2}\overset{\infty}{S}_1\left(\frac{T^n}{m^n n!}\right)\right.$$
$$\left.+2\frac{T}{m}\overset{\infty}{S}_1\left(\frac{T^{n+1}}{m^{n+1}(n+1)!}\right)-2\overset{\infty}{S}_1\left(\frac{T^{n+2}}{m^{n+2}(n+2)!}\right)\right\}$$
$$=\lambda' m\left\{\frac{T}{m}\left(e^{\frac{T}{m}}-1-\frac{T}{m}\right)-2\left(e^{\frac{T}{m}}-1-\frac{T}{m}-\frac{1}{2}\frac{T^2}{m^2}\right)\right\}$$
$$=\lambda' m\left\{e^{\frac{T}{m}}\left(\frac{T}{m}-2\right)+\frac{T}{m}+2\right\}.$$

Accordingly,
$$\bar{t}=m\left\{\frac{e^{\frac{T}{m}}\left(\frac{T}{m}-2\right)+\left(\frac{T}{m}-2\right)}{1+e^{\frac{T}{m}}\left(\frac{T}{m}-1\right)}\right\} \quad \text{.....(xxxviii).}$$

Similarly,

$$N(\sigma_t^2+\bar{t}^2)=\lambda'\left\{\overset{\infty}{S}_1\left(\int_0^T\frac{(T-t)^n t^2}{m^{n+1}(n-1)!}dt\right)\right\}$$
$$=\lambda' \overset{\infty}{S}_1\left\{\int_0^T\frac{(T-t)^n}{m^{n+1}(n-1)!}(T^2-2T(T-t)(T-t)^2)\,dt\right\}$$
$$=m^2\lambda' \overset{\infty}{S}_1\left\{\frac{T^{n+2}}{m^{n+2}}\frac{n}{(n+1)!}-\frac{2T^{n+2}}{m^{n+2}}\frac{n(n+1)}{(n+2)!}+\frac{T^{n+2}}{m^{n+2}}\frac{(n+2)(n+1)^n}{(n+3)!}\right\}$$
$$=m^2\lambda' \overset{\infty}{S}_1\left\{\frac{2T^{n+2}}{m^{n+2}}\frac{1}{(n+2)!}-\frac{6T^{n+2}}{m^{n+2}}\frac{1}{(n+3)!}\right\}$$
$$=m^2\lambda'\left\{\frac{2T}{m}\left(e^{\frac{T}{m}}-1-\frac{T}{m}-\frac{1}{2}\frac{T^2}{m^2}\right)-6\left(e^{\frac{T}{m}}-1-\frac{T}{m}-\frac{1}{2}\frac{T^2}{m^2}-\frac{1}{6}\frac{T^3}{m^3}\right)\right\},$$

or
$$\sigma_t^2 + \bar{t}^2 = \frac{m^2\left\{e^{\frac{T}{m}}\left(\frac{2T}{m} - 6\right) + \frac{T^2}{m^2} + \frac{4T}{m} + 6\right\}}{1 + e^{\frac{T}{m}}\left(\frac{T}{m} - 1\right)} \quad \text{.....(xxxix).}$$

From this we deduce

$$\sigma_t = \frac{m\sqrt{e^{\frac{2T}{m}}\left(\frac{T^2}{m^2} - \frac{4T}{m} + 2\right) + e^{\frac{T}{m}}\left(\frac{T^2}{m^2} + \frac{T^2}{m^2} + \frac{4T}{m} - 4\right) + 2}}{1 + e^{\frac{T}{m}}\left(\frac{T}{m} - 1\right)} \quad \text{.....(xl).}$$

Lastly we may consider the correlation of length of intervals and number of intervals. We have just seen that the regression is not linear in the case of t on n, it would therefore be desirable to have $\eta_{t,n}$ and $\eta_{n,t}$, the two correlation ratios, as well as the correlation coefficient $r_{n,t}$.

To determine the latter we must first find

$$\begin{aligned}
Np_{11}' &= \sum_1^\infty\left\{\int_0^T \lambda' \frac{nt(T-t)^n}{m^{n+1}(n-1)!}\,dt\right\} \\
&= \sum_1^\infty\left\{\lambda' \frac{n^2 T^{n+2}}{m^{n+1}(n+2)!}\right\} \\
&= \sum_1^\infty\left\{\lambda'\left(\frac{T^{n+2}}{m^{n+1}n!} - \frac{3T^{n+2}}{m^{n+1}(n+1)!} + \frac{4T^{n+2}}{m^{n+1}(n+2)!}\right)\right\} \\
&= \lambda' m\left\{\frac{T^2}{m^2}\left(e^{\frac{T}{m}} - 1\right) - \frac{3T}{m}\left(e^{\frac{T}{m}} - 1 - \frac{T}{m}\right) + 4\left(e^{\frac{T}{m}} - 1 - \frac{T}{m} - \frac{T^2}{2m^2}\right)\right\}
\end{aligned}$$

Thus
$$p_{11}' = \frac{m\left\{e^{\frac{T}{m}}\left(\frac{T^2}{m^2} - \frac{3T}{m} + 4\right) - \frac{T}{m} - 4\right\}}{1 + e^{\frac{T}{m}}\left(\frac{T}{m} - 1\right)} \quad \text{.....(xli).}$$

We must now transfer by aid of (xxxvi) and (xxxviii) this product moment to the means. We have

$$p_{11} = -\frac{m\left\{e^{\frac{2T}{m}}\left(\frac{T^2}{m^2} - \frac{4T}{m} + 2\right) + e^{\frac{T}{m}}\left(\frac{T^2}{m^2} + \frac{T^2}{m^2} + \frac{4T}{m} - 4\right) + 2\right\}}{\left\{1 + e^{\frac{T}{m}}\left(\frac{T}{m} - 1\right)\right\}^2} \quad \text{...(xli)}^{\text{bis}}$$

Accordingly we have, by aid of (xxxvii) and (xl),

$$r_{t,n} = \frac{p_{11}}{\sigma_t \sigma_n} = -\frac{m\sqrt{e^{\frac{2T}{m}}\left(\frac{T^2}{m^2} - \frac{4T}{m} + 2\right) + e^{\frac{T}{m}}\left(\frac{T^2}{m^2} + \frac{T^2}{m^2} + \frac{4T}{m} - 4\right) + 2}}{T\sqrt{e^{\frac{2T}{m}}\left(\frac{T}{m} - 2\right) + e^{\frac{T}{m}}\left(\frac{T}{m} + 2\right)}}$$

.....(xlii).

Clearly the regression coefficient

$$r_{t,n}\frac{\sigma_n}{\sigma_t}=\frac{p_{tn}}{\sigma_t^2}=-\frac{1}{m},$$

which agrees with (xxxiv).

It is further easy from (xxxviii) and (xxxvi) to verify that

$$\bar{n}+\frac{\bar{t}}{m}=\frac{T}{m}+1 \quad \text{.....(xliii),}$$

which completes the checking of (xxxiv).

As the mean n_t's for arrays of t are not hard to find*, it would be fairly easy to deduce m from a correlation table by the slope of this regression line. I have not so far succeeded even in this simple case in finding either correlation ratios in a simple form; but as $\bar{t}_n$ and $\bar{t}$ as well as N_n and σ_t are known theoretically, it would be simple to calculate their theoretical values from the given values of m and T arithmetically, using

$$\eta^2_{t,n}=\sum_1^{n_3}\frac{(\bar{t}_n-\bar{t})^2 N_t}{N\sigma_t^2}$$

and to compare this value with that found from the observed value of these quantities. Proceeding in the same way we, of course, find

$$\eta_{n,t}=\frac{1}{m}\left\{S\left(\frac{N_t(t-\bar{t})^2}{N\sigma_n^2}\right)\right\}^{\frac{1}{2}}=\frac{1}{m}\frac{\sigma_t}{\sigma_n}=r_{n,t},$$

or the other η is equal to the correlation coefficient, owing to the linearity of the regression.

(iv) We will now apply the process of finding the frequency of intervals in periods T supposing the periods to contain n intervals. The fundamental formula for this is (xxv)^{bis} which is of the form $f_{n,t}=C_n(T-t-n\beta)^n$, where C_n is a constant dependent on n. The frequencies of lengths of intervals for different values of n are shown in Table III, which is thus a correlation table for the number of intervals in periods of 10 secs. and the lengths of the intervals. No satisfactory method of finding the constants m and β from such a table of observed frequencies has been discovered so we are obliged to use the values found from the frequency of occurrence of the event in period T, i.e. the ratio of frequency method. The P for goodness of fit for the whole table is ·233, but, as in the case of intervals in an indefinitely long time, this low value is due to the bad fit of the first two groups (β to 1 and 1 to 2) for the different values of n. Taking the first group in each case to be β to 2 gives the much improved value $P=$ ·707. The P for the right marginal total of Table III (Fig. IV), that being the distribution of lengths of intervals in 10 secs. irrespective of the number of intervals in that period, when the first group is from β to 2, is ·882.

* The intervals of time, subranges in time, must be small. There is, however, none of the difficulty due to the abruptness and unknown value of β (in the general case) which accompanies the finding of the t_n's.

TABLE III.

$\beta = .354$ $P' = .22$

$M = 2.581$ TABLE OF $P'' = .71$

LENGTHS OF INTERVALS AND NUMBER OF INTERVALS.

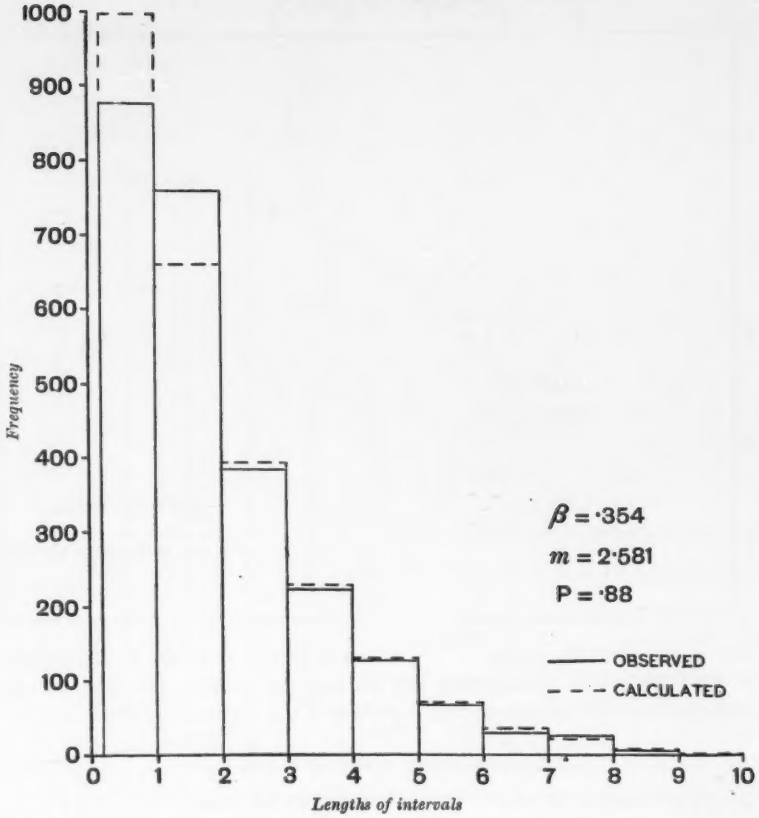
LENGTHS OF INTERVALS	NUMBER OF INTERVALS								TOTALS
	1	2	3	4	5	6	7	8 & OVER	
β - 1	27 25·5	82 93·9	144 158·6	140·5 163·3	103 114·4	48·5 57·8	22 21·8	0 7·9	567 643·2
1 - 2	49 35·9	107·5 118·9	233·5 179·7	182·5 163·9	125·5 100·7	47·5 44·1	16 14·2	0 4·2	761·5 661·6
2 - 3	20 31·5	82 90·4	139·5 116·7	86·5 89·5	39·5 45·3	14 16	4 4·1	0 ·9	385·5 394·4
3 - 4	25·5 27·1	64·5 65·8	75·5 70·5	43·5 43·9	11 17·6	4 4·8	0 ·9	0 ·1	224 230·7
4 - 5	18·5 22·7	45·5 45·1	43·5 38·5	17 18·5	5 5·5	0 1·1	0 ·1		129·5 131·5
5 - 6	20 18·3	23·5 28·3	19 18·1	5·5 6·1	1 1·2	0 ·1			69 72·1
6 - 7	10 13·8	13 15·4	8 6·6	·5 1·5	0 ·1				31·5 37·4
7 - 8	13 9·4	9 6·4	0 1·5						22 17·3
8 - 9	4 5·1	1 1·2							5 6·3
9 & OVER	0 ·6	0 ·2							0 ·8
TOTALS	187 189·9	428 465·6	663 590·2	476 486	285 284·7	114 123·9	42 41·1	0 13·1	2195 2195·3

The theoretical mean length of interval for a given n is shown in the following table together with experimental mean found as for the lengths of intervals in an indefinitely long time.

TABLE IV.

Number of Intervals in Periods of 10 seconds (n)	$\frac{T+\beta}{n+2}$	Experimental Means	β from Experimental Means
1	3·4513	3·3191	negative
2	2·5885	2·6219	·4876
3	2·0708	2·0926	·4630
4	1·7256	1·7903	·7418
5	1·4791	1·4240	negative
6	1·2941	1·2733	·1864
7	1·1504	1·0952	negative

Fig. IV. Lengths of intervals in periods of 10 seconds.

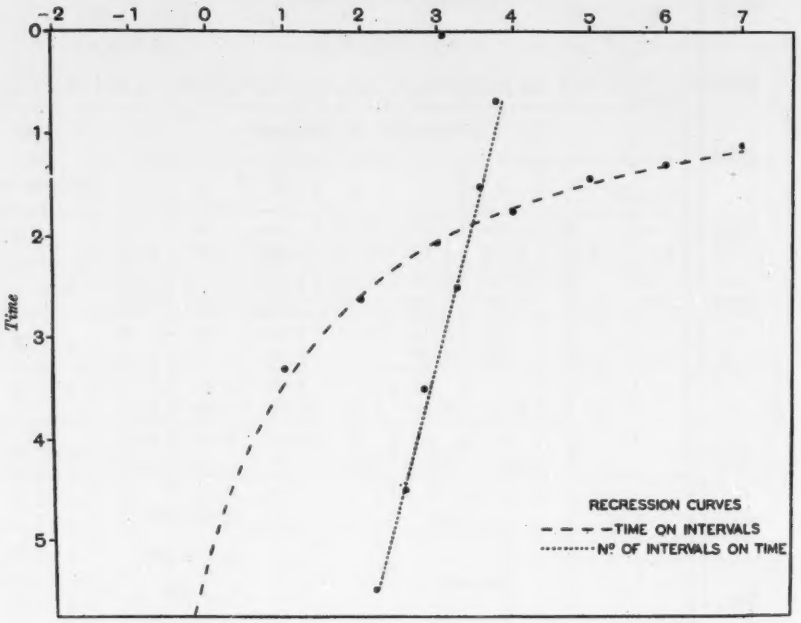


The β's deduced from these latter values are so erratic that it is clear that we cannot hope to get any reliable value from them.

These means are shown plotted in Fig. V, thus providing one regression line of the surface. The other, giving the mean number of intervals for lengths of intervals between t and $(t+dt)$, is calculated theoretically from (xxxi) and it fits the observed means quite satisfactorily; the curve appears to be linear, but we have been unable to deduce a proof even of its approximate linearitv beyond the fact shown above that it is linear for $\beta = 0$.

Fig. VI shows the distribution of lengths of intervals in 10 secs. when there are three intervals in that period; one of the theoretical frequencies being that calculated by $(xxv)^{bis}$ while making the total frequency for $n = 3$ equal to the observed. This method would have to be used if only one column of the table were known.

Fig. V. Number of intervals.



Below are given the values of the constants β and m found by the different methods; it should be remembered that the last two are not found from precisely the same material as the others, only a portion of the whole being used.

Various methods of finding β and m.

A. From duration of intervals in an indefinitely long time.

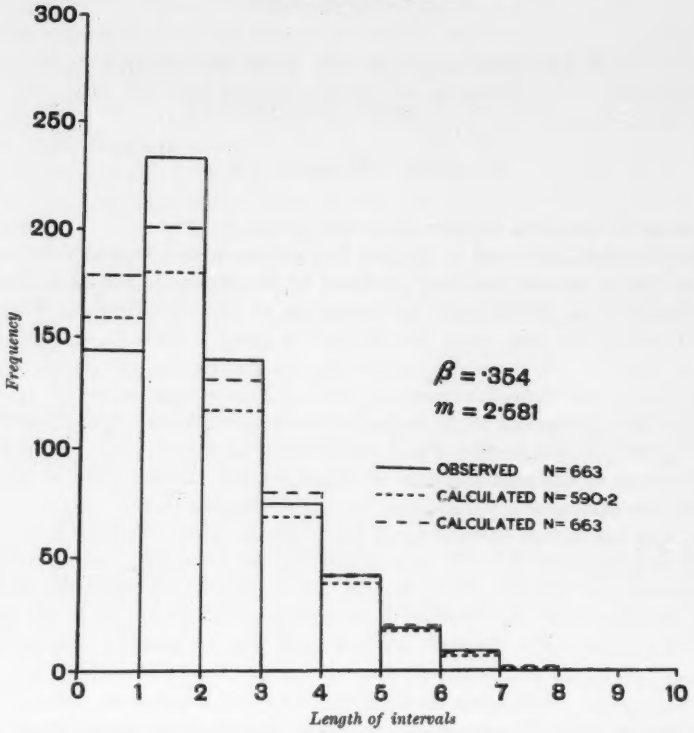
		β	m
(i)	First Group β to ·5	·370,552,	2·488,007,
(ii)	First Group β to 1	·362,918,	2·495,527.

B. From frequency of occurrences in periods T.

		β	m
(iii)	$\chi_n = \left(\frac{n!\, f_n}{f_0}\right)^{\frac{1}{n}}$ method	·353,862,	2·580,641,
(iv)	$\chi_n' = \left(\frac{f_n^2}{f_{n-1} f_{n+1}} \frac{n}{n+1}\right)$ method ...	·393,772,	2·495,831.

These values agree moderately well with each other and we may fairly safely predict that although their probable errors are at present unknown, they will be found small enough to make the methods sufficiently reliable for practical purposes. The tests of goodness of fit are reasonably satisfied and the experimental data may

Fig. VI. Lengths of intervals in periods of 10 seconds when there are 3 intervals in each period.



be held to confirm the above theory of random occurrences, if it is supposed that the discrepancies observed in dealing with short intervals in the neighbourhood of β are due to fluctuations throughout the experiment of the least interval between occurrences. More satisfactory evidence of the value of the theory might be given if experimental or observational data could be found for which β was more adequately constant.

I am indebted to Professor Karl Pearson for suggestion and assistance especially in the algebraic portion of this paper and to Miss Ida McLearn for the preparation of the diagrams.

WAS THE SKULL OF THE MORIORI ARTIFICIALLY DEFORMED?

By KARL PEARSON, F.R.S.

Professor V. Giuffrida-Ruggeri, in a recent paper, has criticised the memoir on the Moriori crania published by Eveline Y. Thomson in this Journal*. It is, perhaps, needless to say that anything produced by the Biometric School is likely to be anathema to an old-fashioned anthropologist of Professor Giuffrida-Ruggeri's type. He views the more exact calculations on which a biometrician would lay stress in much the same spirit as a biometrician regards the old school anthropologist who divides data of a character obtained from various sources for 19 or 20 crania into three groups and thinks a scientific racial differentiation can be based upon comparing the percentages thus found with those of another and similar grouping! The questions of different methods of measurement, the influence of random sampling, the futility of small samples are not difficulties to this type of anthropologist, they are merely shibboleths of the mathematician. Further he is quite sure the mathematician does not look at anything but his figures, and so overlooks the obvious in his material, which he, the old school anthropologist, sees at first glance—not because he has handled more material, but because he is the Simon pure—the genuine anthropologist. Let us admit for the occasion that the mere mathematician might well handle 65 Moriori crania and not find out that that race was in the habit of deforming its skull! But is it not somewhat strange that other observers with at least some claim to be genuine anthropologists—i.e. old-school anthropologists—should have examined Moriori crania and overlooked this deformation?

Welcker† was acquainted with two Moriori crania. He does not refer to any artificial deformation in them, although he does in the case of Peruvian skulls. Flower knew eight Moriori crania and does not refer to their deformation‡, although he was fully aware of the deformation of New Hebridean skulls.

De Quatrefages and Hamy§ handled three male crania of the Moriori and figured one. They do not note any deformation in these crania. They are quite conversant, however, with deformed skulls and figure one. Those who will take

* "A Study of the Crania of the Moriori, or Aborigines of the Chatham Islands, now in the Museum of the Royal College of Surgeons," *Biometrika*, Vol. XI. pp. 82—135. The writer of the paper was not a mathematician but a biologist.

† *Archiv für Anthropologie*, Bd. I. S. 135, 154, 157, etc. 1866.

‡ *Journal of the Anthropological Institute*, Vol. XXVI. p. 295, 1896, and *Catalogue of Royal College of Surgeons Museum*, Part I. Man, pp. 128—130.

§ *Crania ethnica*, *Text*, pp. 460, 461, *Atlas*, Pl. LV. Figs. III and IV.

the trouble to compare the artificial deformed skull on p. 475 of their *Text* with the Moriori skull on Plate LV of their *Atlas* will see at once that the Moriori skull has a naturally receding frontal and not an artificial one; they will see the difference between the post-coronal depression of the Moriori and that of the Peruvian deformed skull. The main difference between Thomson's photograph reproduced (without permission!) by the Italian Professor and Quatrefages and Hamy's lithograph is the tilt in the former produced by adjusting it to the Frankfurt horizontal plane. In the *norma lateralis* of a deformed skull the lowest point of the rim of the orbit joined to the pterion gives a line meeting the posterior part of the sagittal contour in about its most posterior point. In the natural skull it will meet the sagittal contour in the neighbourhood of the apex—i.e. in the *norma verticalis* and not the *norma occipitalis*. See our Plate I.

Fridolin* has described and measured one Moriori adult male cranium. He has not referred to any deformation, yet he was fully conscious of this possibility for he figures and describes a New Guinea skull, "wahrscheinlich künstlich deformirt."

Broesike† has examined two Moriori crania from the Chatham Islands. Of the second he says nothing as to deformation. Of the first he tells us that it is that of a youth, much weathered and very porous with the sutures gaping. He says—which is not to be wondered at—that the skull appears to be misshaped‡ ("verunstaltet"). His description of the asymmetrical "Verunstaltung" accords well with the not unusual post-mortem deformation of the thin cranium of a young person. Anyhow, whether this be the fact or not, he does not describe the skull as artificially deformed, and this is the only reference to deformation of Moriori crania that we have come across in the literature of the subject till Professor Giuffrida-Ruggeri's memoir.

Poll, whose training in the *Anatomische-biologisches Institut* in the University of Berlin should at least have taught him to recognise an artificially deformed skull, had ten male and three female Moriori, with two skulls indeterminable as to sex and five children's crania to work on. He gives almost a page of minute qualitative description to each skull, but he describes none as being "künstlich deformirt." Nor does he tacitly reject lengths or indices which would be meaningless if his crania had been artificially deformed. He discusses them all as racial characters§.

Duckworth, who is not only an anatomist by profession but Lecturer on Physical Anthropology at Cambridge, reported in 1900 on ten Moriori crania at Cambridge‖, but he did more, he examined very carefully for anomalies the 65 Moriori crania at the Royal College of Surgeons afterwards studied by Thomson.

* *Archiv für Anthropologie*, Bd. XXVI. S. 696 and Tafel XIII. 1900.

† *Die Anthropologischen Sammlungen Deutschlands*, Theil Berlin, Erster Theil, S. 51, 52.

‡ He does not use the technical term "künstlich deformirt" but a term which would apply to earth pressure on a thin skull. This is probably the skull which Poll terms Bl. 2 and says is that of a non-adult of 15 years of age; he says it is "porotisch," "zerbrochen" and "eingedrückt," but he says nothing about artificial deformation.

§ *Zeitschrift für Morphologie und Anthropologie*, Bd. v. S. 7, *et seq.* 1903.

‖ *Journal of the Anthropological Institute*, Vol. XXX. p. 141 *et seq.* 1900.

He failed to note any deformation in either the Cambridge or the London series. If the Moriori practised as a race artificial deformation, it entirely escaped him.

Lastly we may note that Scott, the first and one of the best investigators in this field, with a collection of 46 Moriori crania at his command, and acquainted with the practice of deformation in the New Hebrides, as well as close to the men who had studied the Moriori customs at first hand, makes absolutely no mention of his material being artificially deformed!

Up to the time when E. Thomson undertook her examination of the 65 London Moriori crania, nearly 170 Moriori skulls had been handled by anthropologists of considerable distinction, but they had one and all failed to realise that artificial deformation was practised by the Moriori.

Duckworth, I regret, gives no reproductions.

Poll has some excellent plates, unfortunately not orientated to the Frankfurt horizontal*, but they show exactly the same features as Thomson's series, the receding frontal, the pentagonal *norma occipitalis*, the post-coronal depression, the sagittal crest, and the extraordinary massiveness and muscularity of the type.

The mean frontal index† of the male Moriori cranium as given by Thomson‡ is 19·4, while the frontal indices of Poll's Br. 2 and Dr. 1 crania are 18·9 and 19·3 respectively as determined from the photographs, Thomson's 765^{45} has a frontal index of 18·4 of the same order as Poll's Br. 2, or if one skull is artificially flattened, so are the others judged by this test. In Quatrefages and Hamy's illustration—not photographic—this index appears to be 19·4, but it would be difficult to lay stress on what might be very largely influenced by the uncertainties of a lithographed drawing.

Now what scientific line of criticism was open to Professor Giuffrida-Ruggeri if he suspected that the Moriori were a race who practised artificial deformation of the skull?

The best line of conduct would be to have waited before writing his memoir till he had been able to examine at first hand the collections at Bremen, Cambridge or London—I will not suggest that he should have travelled to New Zealand. The next best line would have been to have examined carefully the whole series of photographs—particularly those of the lateral aspect—already published. Thomson gives ten *normae laterales*, Poll two, and Quatrefages and Hamy's drawing might

* Poll writes, S. 124 "Die Photographien der Schädel sind in der halben Grösse angefertigt. Die Schädel sind mit Hilfe der Waldeyer'schen Stativs in die deutsche Horizontale eingestellt; dies gelingt ausserordentlich leicht und bequem." The German horizontal plane should make the plane through the lowest points of the orbits and the highest points of the auricular passages horizontal. It is hard to imagine this is so in Poll's *norma lateralis* in Fig. 3, still less in Fig. 6 of Tafel II. We think there must have been some error of orientation either in the original photograph, or in the trimming of the border by the engraver to make these photographs fit his plate. But the question of the tilt has an all-important bearing on the aspect of the *norma lateralis* and influences the judgment of a superficial observer of the photograph as to whether a cranium looks "deformed."

† Frontal Index = 100 × subtense of nasio-bregmatic arc divided by the nasio-bregmatic chord.

‡ *Biometrika*, Vol. xi. p. 95.

be taken into account. If, after doing this and comparing these representations with any long typical series of Peruvian and New Hebridean skulls, he had not concluded with anatomists and anthropologists like Scott, Poll, Flower, Turner and Duckworth, that there is no question of deformation—I will not suggest that he should have analysed numerically the extraordinary differences in absolute lengths as well as indices between deformed and natural series—he should have criticised in the first place the obtuseness of these anatomical craniologists* who could overlook such a fundamental point!

Professor Giuffrida-Ruggeri does not adopt such a course. He takes *one* out of *ten* of Thomson's *normae laterales* and says this skull is artificially deformed *ergo* the series as a whole is artificially deformed, *ergo* all these elaborate biometrical investigations here (and no doubt elsewhere) are idle. To point the moral as strongly as possible, the inference drawn from a single photograph of a single skull is carried to a sweeping conclusion in the title of the Professor's memoir. It runs "A proposito della leptorrinia dei Moriori e della loro deformazione cranica†." On such a slender basis the Moriori race, according to Professor Giuffrida-Ruggeri, henceforth is to be looked upon—not as one with most noteworthy cranial characters—but merely as a race with a self-distorted cranium.

He does indeed say that Poll—without suspecting the deformation—has drawn attention to the markedly receding frontal, but this had also been done by Scott and Duckworth. Thomson, however, was the first to show that the Moriori have a smaller frontal index than any other measured race, and this is how Giuffrida-Ruggeri sums up the biometric work on the point after referring to p. 95 of her memoir:

> Altri calcoli piu precisi—ma inutili—si leggone a p. 130. Anche a p. 110 altre valutazioni dell' appiattimento dell' osso frontale—molto complicate, ma fuori di posto, essendo viziate della stessa inavvertenza morfologica—danno risultati sfavorevoli ai Moriori e non potrebbe essere diversamente (p. 7 of offprint).

Why indeed must the Moriori be an artificially deformed race? Simply because a flat frontal is "not a character of the South Oceanic races," and therefore its very existence must be interpreted as a deformation. But this is to beg the fundamental question of whether the Moriori were a "South Oceanic race" or whether, pure or hybridised, they were a still earlier race being pushed to the extremes by a South Oceanic racial invasion. The question of deformation has to be settled quite apart from whether a flat frontal is or is not a character of the "South Oceanic races." And it has to be settled appreciatively or quantitatively by comparing a series of the Moriori against a series where deformation is admitted to be practised. Postponing for a time the question of quantitative measurement‡, I examined the Peruvian and New Hebridean series in the Royal College of Surgeons Museum as a whole against the Moriori series, and have no doubt that the Moriori as a race

* More than one of them had actually examined in whole or part the series studied by Thomson!

† *Rendi della R. Accademia delle Scienze Fisiche e Mathematiche di Napoli*, Ser. 3ª, Vol. XXVII, 1921.

‡ For example the cephalic index of the male Moriori is 76·15 with a standard deviation of 2·58, but that of 47 Peruvian male crania is 89·15 with a standard deviation of 8·25!!

at the time these crania were those of living persons did *not* practise deformation. I think this is definitely indicated not only by the bulk in all aspects of Thomson's plates, but by her figures, in particular her mean sagittal contours (pp. 113 and 115 of her paper). It is also indicated by Poll's cranial photographs and photographs of living representatives of the Moriori*. However, some general impression of what a deformed skull looks like and what it does not look like may be obtained if the reader will examine the accompanying plate in which I have reproduced the *normae laterales* of two specimens practically taken at random from each of the Moriori, New Hebridean and Peruvian series. Apart from this I think the pentagonal *norma occipitalis*, the curvature of the posterior portion of the sagittal contours, the sagittal crest and the extraordinarily rugous and muscular character of the whole skull are quite sufficient to put deformation out of the question.

Is the skull 765[40] singled out by Professor Giuffrida-Ruggeri really a case where there may have been some deformation? I personally think not, it only shows the racial characters of the Moriori in a possibly somewhat emphasised manner†. If the view be taken that the bulk of the Royal College series of crania are normal and this particular one artificially deformed, then the continuous graduation of the frontal index throughout the series will have to be met. And further, are we to term artificially deformed all seventeenth century English crania —and they are fairly numerous—with a lower frontal index than the mean Moriori skull? Plates XII, XIV and XVI of *Biometrika*, Vol. V exhibit such crania. I think there is very little doubt that the anatomical anthropologists will agree that the artificial deformation of the Moriori has remained undiscovered up till to-day—simply because it does not exist. The Moriori have a minimum frontal index, not because they deformed their skulls, but because they contained a racial element more primitive than the bulk of the "South Oceanic races"‡.

* These photographs unfortunately in no case give a profile, and it is accordingly difficult to appreciate the retreating frontal. Giuffrida-Ruggeri dismisses these photographs by saying that the later Moriori may well have given up the practice of deformation. But our type contours do not suggest bad fits to the living head.

† Professor Giuffrida-Ruggeri wrote apparently to Sir Arthur Keith asking him to verify—not whether the Moriori series was as a whole artificially deformed but—whether 765[45] was artificially deformed "come io sospettaro." According to Professor Giuffrida-Ruggeri the reply fully confirmed the view that it is a case of artificial deformation. I hardly interpret Sir Arthur Keith's reply to sanction the hypothesis of a *general artificial deformation* of the Moriori crania. He wrote: "I have just examined the skull Chatham Islander 765[45] and compared it with deformed skulls from the New Hebrides. You are right! it shows exactly the same frontal features as the deformed Hebridean—differing from them in the great parietal width." While it shows as well as other Moriori crania do a flat frontal, a character of the Hebridean deformed crania, they show entirely different features not only of the frontals, but of the parietals and occipitals from those of the New Hebrides, the features in the latter case being markedly correlated with the artificial frontal changes.

‡ Professor Giuffrida-Ruggeri speaks of the postbregmatic depression ("insellatura retrobregmatica"), noted by both Poll and Thomson, as in his opinion evidence of artificial deformation. We found it equally conspicuous in many seventeenth century English crania, but were unable to associate it with any "legatura trasversale al vertico," which we could trace in the customs of the day; it occurred in over 19% of 412 English crania. It can therefore hardly be regarded as in any way evidence of artificial deformation.

Passing from this fundamental conception of our Italian professor—i.e. that the Moriori only differ from other "South Oceanic races" because they deformed their skulls—we turn to his discussion of the nasal index. We have seen that he is content to divide nasal indices into the percentages falling into three classes and does not trouble himself about the probable errors of these percentages even when based on 19 crania. He compares Poll's distribution with Thomson's and seems suspicious of the latter's results, because her differences for the two sexes are very considerable. Now Thomson states very clearly how she measured her nasal height, i.e. from the nasion to the lowest point of the edge of the left pyriform border, or if this were damaged, to the lowest point of the edge of the right aperture. Poll says he has taken his measurements according to the Frankfurt Concordat. He may have measured as Thomson did, but actually that "Verständigung" runs on this point as follows: "*Nasenhöhe*, Fig. 2, wNH: von der Mitte der Sutura naso-frontalis bis zur Mitte der oberen Fläche des Nasenstachels resp. zum tiefsten Rand der Apertura pyriformis" (*Archiv für Anthropologie*, Bd. xv, S. 4, 1884). This would justify a measurement to the nasal spine. It is therefore not absolutely clear what Pöll may have done. But, in the results cited by Giuffrida-Ruggeri, Poll has pooled with his own data results obtained by Scott, Turner and others who certainly did not know anything of the Frankfurt Concordat *. It is not therefore justifiable to place Thomson's distribution against Poll's without further consideration of this point. The sexed crania of Duckworth and Poll are so few that the means are subject to large probable errors, but it is still suggestive to consider the differences in the results reached by various authorities. The numbers upon which each depend are given in brackets in the following Table.

Measurement of the Nose.

	Males			Females		
	Nasal Height	Breadth	Index	Nasal Height	Breadth	Index
Poll	54·4 (6)	26·8 (6)	49·1 (6)	50·4 (5)	26·3 (5)	52·0 (5)
Scott	57·5 (32)	26·5 (32)	46·1 (32)	50·9 (10)	24·8 (10)	48·8 (10)
Thomson ...	57·3 (34)	25·3 (34)	43·9 (34)	52·5 (21)	25·3 (21)	48·2 (21)
Duckworth ...	56·2 (6)	24·2 (6)	43·8† (6)	52·3 (3)	24·3 (3)	46·5 (3)

* Perhaps the most illusory description of nasal height is that given by Sir W. H. Flower (Royal College of Surgeons' Catalogue, Man, p. xviii), "A vertical line between the nasion and lower border of the nasal aperture." The height will not be "vertical" in any case. Flower's figure (p. xii) shows he did not intend it to be vertical. If he meant in the vertical or sagittal plane of the skull then the measurement must be to the nasal spine, rather than to the border of the pyriform aperture. If he meant to the lowest point on the border of the pyriform aperture, then his nasal height is neither vertical, nor in the vertical sagittal plane, and his figure can only represent not the real length but a projection of the real length! Unfortunately many English craniologists have only had Flower's definition before them.

† Duckworth himself gives for his mean index for *seven* male crania 44·3 (p. 145) but his table p. 150 shows only seven male crania in all and one has no nasal measurements.

In all these cases we see that the male is markedly more leptorrhine than the female*, and that Thomson's mean for males agrees with Duckworth's and her mean for females with Scott's. Her results differ widely from Poll's, but so do Duckworth's and, to a considerable but lesser extent, Scott's. It seems to me that if the paucity of numbers be not regarded (the probable error of Poll's ♂ mean is about 0·83), we must conclude that Poll measured his nasal height somewhat differently to Thomson, and accordingly Professor Giuffrida-Ruggeri is not justified in contrasting his results directly with hers.

If, instead of working with means, we use the loose threefold category apparently approved by Professor Giuffrida-Ruggeri, we find:

Distribution of Nasal Indices in Percentages.

Author...	Male				Female			
	Poll	Scott	Thomson	Duckworth	Poll	Scott	Thomson	Duckworth
Leptorrhine	0·0	59·4	88·3†	83·3	20·0	20·0	35·7†	33·3
Mesorrhine	100·0	31·2	11·7	16·7	20·0	60·0	31·0	33·3
Platyrrhine	0·0	9·4	0·0	0·0	60·0	20·0	33·3	33·3
Number	6	32	34	6	5	10	21	3

Results at once flow from this table which bear strongly on Professor Giuffrida-Ruggeri's classificatory table (p. 4, *loc. cit.*). In the first place Duckworth's own measurements absolutely confirm Thomson's, although with his few cases it was hardly to be expected that so close an agreement would arise, and it must be of course somewhat fortuitous. Poll's results differ widely from those cited by Giuffrida-Ruggeri from Poll, simply because he has taken from Poll the data Poll gives for

* This sexual difference is fairly general. Thus we have:

Nasal Index.

	Naquada	Modern Egyptian	Alt Bayerisch	Aino	Würtemberger
♂	51·1	49·3	48·7	50·7	46·6
♀	52·3	54·6	49·2	50·5	50·3

	17th Century English	Maori	Congo Bantus	Long Barrow	Burmese
♂	47·6	47·9	55·2	49·0	52·8
♀	47·8	49·1	55·8	49·1	52·9

	Kareens	Malayans	Chinese	Hindus
♂	46·4	50·4	48·9	49·7
♀	50·4	53·6	48·1	50·4

and the general tendency is for the female to be more platyrrhine than the male.

† These results do not accord with those Professor Giuffrida-Ruggeri attributes to Thomson. Thomson has only 21 available nasal indices for females, not 22.

his own measurements *combined* with other craniologists' results. But, as we have already stated, Poll's data show that he did not probably take the nasal height in the same way as Thomson. If we suppose him to have measured to the nasal spine this will be about 5% shorter than Thomson's measurement (·9485 and ·9595 factors respectively for ♂ and ♀). We then obtain results:

	Poll	
	♂	♀
Leptorrhine ...	50·0%	20%
Mesorrhine ...	50·0%	40%
Platyrrhine ...	0·0%	40%

which more nearly approach those of other writers, especially when we consider the smallness of the numbers Poll had at his command.

If we turn to Scott we find that in his memoir he measured for nasal height the nasio-spinal length, i.e. he did not take the length to the lowest point of the pyriform border. He also classified leptorrhine, mesorrhine and platyrrhine according to Flower* and did not use the *Frankfurt Verständigung* divisions. Thus his nasal index and statements as to class of nasal index are not directly comparable with Thomson's. Accordingly they ought not to be compared directly with Thomson's results; it is difficult, however, to correct for the difference between nasio-spinal and nasio-pyriform border heights in the case of Scott's males, for Thomson's and his mean values of the heights are here the same. If we assume that the multipliers for the indices may be given the values found above—say ·95 and ·96 in round numbers for male and female respectively,—we obtain

	Scott	
	♂	♀
Leptorrhine ...	84·3%	50%
Mesorrhine ...	15·7%	40%
Platyrrhine ...	0·0%	10%

Thus Scott falls quite reasonably into line with the results of Thomson and Duckworth, and all three show precisely the sex difference to which our Professor takes exception. The differences in the matter of nasal index in the results of various writers, over which Professor Giuffrida-Ruggeri seems to be puzzled, could have been largely cleared up had he investigated whether the writers had measured the same character or classified its values in the same way.

* Leptorrhine below 48, mesorrhine from 48 to 53, platyrrhine above 53. This is very different from the 47·0 and lower, 47·1—51·0, and 51·1 and above of the Verständigung.

According to Professor Giuffrida-Ruggeri, the leptorrhiny, the receding frontal and the massiveness and vertical ramus of the mandible of the Moriori are not characteristics marking them off as an extremely primitive race, but the product of climate, artificial deformation and diet. In short, are due to environmental conditions, which also in part have influenced Fuegians and Eskimos, and thus account for various common features of these "fringe" races. He does not explain whether this adaptation is due to long ages of selection by which the skull has retrogressed in many respects to a more primitive form, or is due to the inheritance of acquired characters, in which case the flat frontal ought to have become a permanent racial character and would not need to be artificially deformed. If we prefer th iew after a close examination of the crania—which is an advantage Professor Gi -Ruggeri has not had—and after a careful measuring and analysis of the measurements to look upon this race as forming one of the most primitive that have survived to recent times—and not as a product of environment*, does this justify Professor Giuffrida-Ruggeri in stating that:

La Thomson, essendo della scuola del Pearson, naturalmente non s' interessa del materiale umano che nel senso biometrico (p. 9).

We can assure the worthy professor that from its founder, Francis Galton, downwards our school did not take up anthropology because its members were interested in biometric methods, but because, studying "human material," they found real progress was impossible unless new methods were invented and applied. And it is because of that desire to understand human development that they do not hesitate totally to condemn as nugatory and profitless the descriptive and appreciative processes in which Professor Giuffrida-Ruggeri puts his trust.

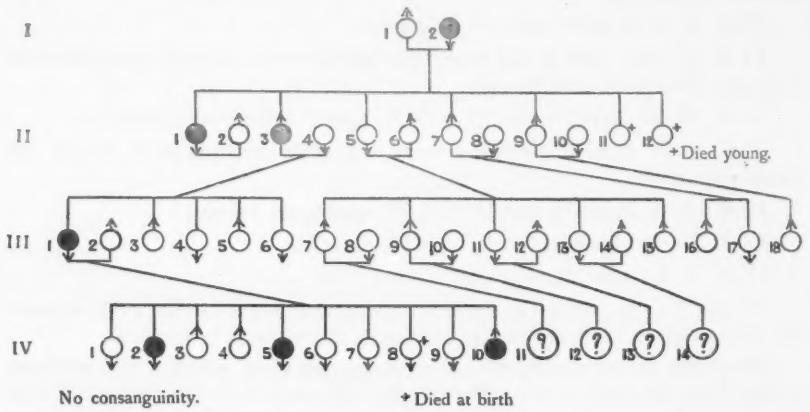
* Or as, "un singolare prodotto della domesticazione"!! (p. 9).

A NEW WHITE-LOCK FAMILY (BARTON'S CASE).

By KARL PEARSON, F.R.S.

The hereditary nature of the "flare"—or pigmentless forehead patch accompanied by a white or yellow white-lock of hair—in man is now well known. Several illustrative pedigrees were published by Nettleship, Usher and Pearson in their *Monograph on Albinism**, and a good pedigree was also published by Dr E. A. Cockayne in 1914 under the title of "A Piebald Family†." That family embraced six generations of the hereditary flare. In the present case we have only been able to trace four generations with a total of seven affected members. Dr Cockayne's pedigree involved nineteen affected members, while Rizzoli's pedigree of the Bianconcini family extends to six generations with twenty affected members. I owe the case to Dr E. R. Barton, of University College Hospital, who came across III. 1 and IV. 10 in the Midwifery Department of the hospital and most courteously put me in touch with the family. The pedigree should accordingly be referred to under Dr Barton's name.

There is no consanguinity. The family goes grey early and there has been a certain amount of tuberculosis.



I. 1. T. H. Normal, died at about 40.

I. 2. J. F. Married I. 1 and is the earliest known ascendant to have the flare. She had both white forehead patch and white lock. She died at about 80.

* *A Monograph on Albinism in Man*, Part I, p. 254, Cambridge University Press.

† *Biometrika*, Vol. x. pp. 197—200, with eight plates.

II. 1. R. H. White forehead patch and white lock. She married but has had no children.

II. 3. E. H. Married L. L. (II. 4). E. H. died at about 40 and had white forehead patch and white lock; not known whether he had any body patches.

II. 5. J. H. Normal, married II. 6, normal. She has had five normal children. One of these, III. 15, died at 25, the other four III. 7, 9, 11 and 13, all married and had again normal families, IV. 11, 12, 13 and 14.

II. 7. H. H. Normal, married a normal woman, II. 8, and had a boy and girl, III. 16 and 17, both normal.

II. 9. A. H. Normal, married a normal woman, II. 10, and has had a normal son, III. 18.

II. 11 and 12 died young, but are said to have been normal.

The family of II. 3 consisted of three daughters and two sons of whom only one was affected.

III. 1. R. J. H. Has a yellow-white lock and white forehead patch, said to have no white body patches. She married III. 2, W. S. R., a normal.

III. 3. A. W. H. Died owing to an accident at 24—25 years. He was normal.

III. 4. L. H. Was normal and died at 8 years, of meningitis.

III. 5. Normal, died as a baby.

III. 6. D. H. Normal, is alive at 28 years and unmarried. Thus the trait could only be carried on through III. 1. She has so far had only ten children, but three of them are affected.

IV. 1. R. R. Now 18 years old, is normal.

IV. 2. L. C. R. Died at 15½ years. She had the white forehead patch, the white lock and white patches on the skin.

IV. 3. W. R. Aged 14, and IV. 4, G. R., aged 11½, are both normal.

IV. 5. A. R. Aged 9, has yellow-white lock, white forehead patch, but no skin patches elsewhere.

IV. 6. D. R. Aged 7½, and IV. 7, G. R., aged 5, are normal.

IV. 8. Died at birth, and nothing is known of her.

IV. 9. B. R. Aged 1½, is normal.

IV. 10. T. C. R. Aged four weeks when seen, has quite a marked forehead patch and white lock on his fairly profuse dark hair. No white body patches.

There are three marriages—non-consanguineous—of affected with apparent normals, each of which produce affected. The affected in each case produce affected. The character therefore cannot be recessive. We are compelled to treat it as dominant, but as I. 2 has some normal offspring, she must have been heterozygous. It will be clear that none of the affected can be looked upon as pure dominants, and in their families there ought to be 50 °/₀ affected.

We must leave out II. 11 and II. 12 as there is no record, and IV. 8 who died at birth although III. 1 considered the infant normal. We have accordingly 2 in 5,

1 in 5 and 3 in 9 in generations II., III. and IV. affected or 6 in 19. This is a fair approximation to a third, but rather a poor one to a half. There is in fact a deviation of 3·5 with a standard deviation of $\sqrt{19 \times \frac{1}{3} \times \frac{1}{2}} = 2{\cdot}179$, or the deviation is 1·61 times the S.D. The odds against such a defect are more than 17·5 to 1. Of course not impossible but improbable.

On the other hand there is no case in which an apparent normal marrying a normal has transmitted the character. In the case of IV. 11, 12, 13 and 14 there are considerable families, exact numbers not available, but it is known that none are affected. Thus it would appear that transmission through the unaffected either does not ever, or at least commonly, occur.

This rule holds also for Cockayne's family and for the Bianconcini.

IV. 2 is the only one with body pigmentless patches. As III. 1 mentioned this fact without special enquiry, I think we must take it that her statement with regard to IV. 5 and IV. 10 and to herself that they and she have no pigmentless body patches can be accepted. She believed that II. 1 had no such patches, but did not know about II. 3 or I. 2. It would seem therefore that in this family at least conspicuous body patches are not frequent. It should not therefore be looked upon as a piebald family in the ordinary sense. It exhibits the inheritance of a "flare."

ON THE SESAMOIDS OF THE KNEE-JOINT.

By KARL PEARSON, F.R.S. and ADELAIDE G. DAVIN, B.Sc.

PART II. EVOLUTION OF THE SESAMOIDS.

	PAGE
(8) Further Remarks on Origin of Sesamoids	350
(9) Carnivores	351
(10) Ungulates	353
(11) Pinnipedia	355
(12) Chiroptera	355
(13) Insectivores	356
(14) Rodents	357
(15) Primates	360
(16) Edentates	366
(17) Marsupials	371
(18) On Changes in the Muscular Attachments of the Knee-Joint	376
(19) Monotremes	383
(20) Centre of Ossification of Fibular Crest in Monotremes	384
(21) Birds	387
(22) Reptiles	389
(23) Amphibia	391
(24) Conclusions	392
Description of Plates (Part II)	396

(8) *Further Remarks on Origin of Sesamoids.*

In the first part of this paper attention was drawn to Gruber's statement that the sesamoid invariably arises from hyaline cartilage. We cited our own work as supporting this view and a considerable number of further sections made since entirely confirm it. We do not think it needful to publish the drawings made from these sections—they all show the same hyaline structure; and we now hold with Gruber that no thickening in the tendons of *M. gastrocnemius* or *M. popliteus* should be classed as a hemisesamoid unless it shows this hyaline structure*. Every

* Another matter of some interest is elucidated by our sections. Lunghetti (*Internationale Monatschrift für Anatomie und Physiologie*, Bd. XVI. § 47) following Gillette in his classification of sesamoids into peri-articular and intratendinous claims to have shown that the origin of the peri-articular sesamoids is totally different from that of the intratendinous. We have already (p. 156) criticised Gillette's classification; it does not seem to us in accordance with the evolutionary origin of the *fabella lateralis* and the *cyamella*, both are ultimately peri-articular in origin, and our sections seem to show that Lunghetti's view is erroneous. He asserts that the peri-articular are prefigured by hyaline cartilage which can be noticed in the foetus with the form of the definite bone. With this we are in agreement; the sections of the metatarsal sesamoid in the human foetus (p. 159) demonstrate it. Lunghetti further asserts that the intratendinous only appear in the adult in the middle of a fibrous tissue which chondrifies a short time before ossification. This assertion may be true of pseudo-sesamoids such as the osteomata of riders. It is entirely erroneous in the case of *fabellae* and *cyamella* which can be detected in embryos and the young shortly after birth by the presence of patches of hyaline cartilage as we have indicated: see our pp. 158—159 and Plates XVII and XVIII.

orthosesamoid will be found to exhibit this prefiguration if the knee-joint be examined in the embryonic or early post-natal states.

Two points of much bearing on what follows later were reached as the outcome of these investigations:

(*a*) The *cyamella* of the rabbit was found near the head of the fibula, there was no marked popliteal groove for it to lie in.

(*b*) The lateral and mesial *fabellae* while both prefigured by hyaline cartilage in kittens were very unequal the former being far more developed. Pfitzner* says that he found the *fabella medialis* absent in 29 cases out of 52 in the cat. He states that it always occurs in dog, fox and hare. Meckel† states that both in cat and dog only the external *fabella* occurs. Gruber‡ places *Canis familiaris* (as well as *C. vulpes* and *C. lupus*) and *Felix domesticus* in the group which have both *fabellae*. Thus he tells us that in puppies at birth he found hyaline cartilage and later orthosesamoids; the external ossified first at about six months. The external *fabella* articulated with the femur in a trough ("Grube") with projecting rim on the upper part of the external condyle of the femur; the internal *fabella* was in a less marked trough on the upper part of the internal condyle. With regard to newborn kittens both *fabellae* were prefigured by hyaline cartilages: the external ossified first and the internal was smaller and of a different shape.

(9) *Carnivores.*

While our experience with kittens accords with that of Gruber, and not with that of Pfitzner, and while we have found the two *fabellae* in many adult dogs, we have not invariably found the mesial *fabella* after most careful sectioning of the tendon of the internal head of *gastrocnemius* in puppies of a fortnight old (failed in two cases). On a general balance of the evidence therefore we are inclined to believe that cases do occur in the smaller carnivores, in which both *fabellae* are not present, although Meckel's view that only one occurs in dog, cat and fox is certainly incorrect.

While dealing with the carnivores we may note that Gruber examined a female, aged one year, of *Felis leo*, and found an external hemisesamoid, the internal was represented by a pad ("Wulst") with no cartilage cells (*loc. cit.* p. 59). Further in *Gulo vittalus* there was an external *fabella* only, which as in man did not articulate with the femur, but lay in the tendon of *gastrocnemius*, and between that and the knee capsule (p. 57). Further the external *fabella* and that only has been found by Gruber in *Nasua*, *Ursus arctos*, *Mustela alpina* and *Lutra marina*. Blainville found none in *Ursus*, Meckel found the external only. The latter found the external only in *Felix lynx*, the raccoon, and the hyaena§, while Blainville and Owen found both in the hyaena; and Meckel found both in *Mustela* and *Lutra*. Davis‖ and

* See work cited (*Biometrika*, Vol. XIII. p. 163), S. 582.

† *System der vergleichenden Anatomie*, S. 634, 635, Halle, 1828.

‡ See work cited (*Biometrika*, Vol. XIII. p. 156), pp. 57, 58.

§ *Loc. cit.* S. 634—635.

‖ *Journal of Anatomy and Physiology*, 1888, p. 215.

again Young* found the external *fabella* only in *Viverra civetta*†; and Macalister found none in the Laughing Otter (*Aonyx*)‡.

Lack of material has hindered our undertaking a considerable number of independent dissections on the carnivores; we have been largely compelled to examine prepared skeletons, and these can never prove a negative. They may, however, give positive evidence, and often confirm or refute earlier statements. The chief difficulty is one we have already referred to—the articulator preserves the *fabellae*, but far more rarely, when it exists, the *cyamella*.

We have found both *fabellae* and *cyamella* in: *Proteles cristatus* (Aard-Wolf), *Canis vulpes* (fox), *Canis domesticus* (not invariably) and *Felis domestica*§ (invariably: see Plate XX, Fig. 44).

We have found both *fabellae*, no *cyamella* in *Felis catus* (wild cat), the two-spotted Paradoxure, the *Hyaena crocuta* (spotted Hyaena), *Otocyon lalandii* (long-haired fox), the Java otter, *Cryptoprocta ferox* (fossa) where the external was much larger than the internal, *Felis nebulosa* (clouded leopard), *Canis melitus* (American wolf), *Cyon sumatrensis* (Malayan wild dog).

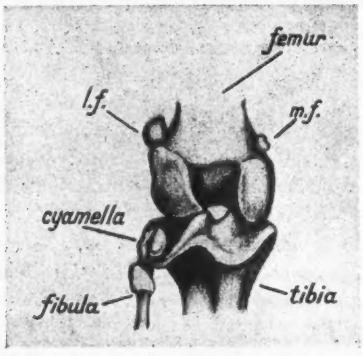
We have found the external *fabella* only in *Lutra canadensis* (Canadian otter), *Aelurus fulgens* (Panda), *Lutra vulgaris* (otter), *Enhydra marina* (sea otter), where the external *fabella* was markedly large and there was a much emphasised excavation corresponding to it above the external condyle on the shaft of the femur itself (see Plate XIX, Fig. 40), *Felis tigris* (2 cases)—*Eupleres goudoti* (a Madagascar

* *Journal of Anatomy and Physiology*, 1879, p. 174.

† Macalister (*Proc. Royal Irish Acad.* Vol. 1. 2nd series, p. 512), states that there is a hemisesamoidal *cyamella*.

‡ *Proc. Royal Irish Acad.* Vol. 1, *Science*, 2nd Series, p. 545.

§ The cat is a very good example of the complete system in the carnivores (see accompanying drawing after Plate XXV, Fig. 6 of Pfitzner's memoir). In it we see the intimate relation of the *cyamella* to



both head of fibula and head of tibia, with the latter of which it articulates. The close association of external *fabella* with the *cyamella* when the limb is flexed is evident and also the dominance of the former over the mesial *fabella*.

viverrid), cheetah (2 cases), *Herpestes ichneumon* (the Egyptian ichneumon) with both lateral *fabella* and *cyamella*, *Canis dingo**.

We have not been able to confirm Gruber's result for the lion or Meckel's for bears. In *Felis leo* we have found a deep cavity on the articular surface of the external condyle which might well be for an external *fabella*. On a skeleton of *Ursus horribilis* (grizzly bear) we found nothing, but on the fossil *Ursus spelaeus* (great cave bear) there were depressions on the femur which might well have served for *fabellae*. The same remark applies to the fossil *Arctotherium bonariensi*, which we also examined.

We have purposely discussed the carnivores in the first place as a group marked by sesamoidal characters intermediate between the Primates and Ungulates. The higher primates have lost wholly or in part both *fabellae* and *cyamella*; the higher carnivores are in the same stage. Either they have never developed them, or they have lost them and comparison with the primates suggests that the latter is the better account. But one great feature comes out of any examination of the carnivores: There is no single species in which the mesial *fabella* is present without the external. If one *fabella* alone occurs, it is the external. The mesial *fabella* seems to be a pale reflex of the external *fabella*. It is often absent, often much smaller, oftener remains a hemisesamoid, and when it does ossify, ossifies later. Its development in the individual suggests a different evolutionary origin, and one of less importance than that of the external *fabella*.

(10) *Ungulates.*

Let us now pass to the Ungulates and note first what others have observed. Among the true ungulates, the *Perissodactyla*, tapirs, horses, etc., no *fabellae* have been seen. Among the *Artiodactyla* the swine† are without them and nearly all the ruminants. The only exception that we have seen noted is that of the stag, which is said by Meckel to have an external *fabella*. This is supported by Gruber, who says he found the external *fabella* in *Cervus tarandus* but otherwise in none of the ruminants. It fails in other members of the *Ungulata*, horse, elephant, hippopotamus, rhinoceros, giraffe, etc., etc.

While we have found sesamoids preserved in mounted skeletons of nearly all the lower primates, the rodents and certainly fifty per cent. of the carnivores, we have drawn a complete blank with the exception of *Hyrax* in the case of all ungulates. There seems no reason why the articulator should have made a practice of removing the *fabellae* in the case of the ungulates.

We have examined skeletons of: *Ovis aries*‡, *Bos indicus*, *Capra hircus* (common goat), *Capra tartarica* (Saiga antelope, 3 cases), *Tetraceros chiliana* (four horned antelope, 4 cases), *Moschus moschiferus* (musk deer), *Cervus dama* (fallow deer),

* Only one case, which had solely the external *fabella*. But the dogs are very difficult unless directly dissected. We have noted both *fabellae* in articulated skeletons of Saint Bernard, French Bloodhound and Chow; external only in Bulldog, Newfoundland and Deerhound, but this proves nothing.

† Gruber found a pad (Wulst) in *Sus scrofa* in the external head of *gastrocnemius* but no hemisesamoid, much less an orthosesamoid.

‡ Several knee-joints of sheep on dissection showed no traces of *fabellae* or *cyamella*.

Rangifer tarandus (reindeer, 2 cases), *Tragulus javonicus** (moose deer, 3 cases), *Cervus elaphus* (red deer), *Connochaetis gnu* (white tailed gnu), *Alces machlis* (moose or elk), *Cervus davidianus* (David's deer), *Cervus puda* (Puda deer), *Hyomoschus aquaticus* (African chevrotain), *Cervus capreolus* (roe, 2 cases), *Hydropotes inermis* (Chinese water deer), *Strepsiceros kudu* (koodoo), *Auchenia pacos* (alpaca), *Auchenia vicugna* (vicugna), *Sus scrofa* (Indian wild boar, 3 cases), *Dicotyles labiatus* (white lipped peccary, 2 cases), *Phacochaerus africanus* (Aelian's wart hog), *Hippopotamus amphibius* (2 cases), Baird's tapir, *Equus zebra, Equus asinus, Camelus bactrianus, Camelopardalis giraffa* (Nubian giraffe), *Elephans indicus, Rhinoceros unicornis, Rhinoceros sumatrensis, Bos taurus* and also *Equus caballus*, Orlando (the thoroughbred race-horse) and numerous other ungulates.

In the face of the large number of skeletons of *Cervidae* examined, it seems hard to believe that the external *fabella* has been removed in all these cases, i.e. that it can be, as Meckel asserted, universal in *Cervus*†.

We have dissected a specimen of a young *Tragulus*, probably from Malasia, but found no *fabellae* nor indeed any sesamoids of the knee-joint. In our opinion an indication of the ancestral history.

Apart from *Cervus*, *Hyrax* is the only ungulate with sesamoids of the knee-joint. According to Blainville both *fabellae* are present, but Meckel and Brandt credit it only with the external *fabella*. We first examined two skeletons. On one *Hyrax dorsalis*, there was no trace of a *fabella* left, if there ever had been any; on the other *Hyrax capensis*, there was a small external *fabella*. Murie and Mivart in their "Myology of *Hyrax capensis*‡" describe *gastrocnemius, plantaris* and *popliteus*, but do not refer to any sesamoids. We were able to examine further one knee-joint of *Hyrax* with certain of the muscular attachments (see Plate XX, Fig. 45). Unfortunately the popliteal tendon had been cut short before it passed over the fibula so it is impossible to say whether a *cyamella* ever existed. The external *fabella* was present, but *gastrocnemius* had shrunk in drying, so that the sesamoid was lifted off the external condyle. There is a slight depressional facet on the articular surface of the internal condyle suggesting the possibility of a mesial *fabella*. There is, however, no doubt about the existence of the external *fabella*. In another mounted specimen of *Procavia*, we have found not only the external *fabella*, but an external anterior *lunula*. These facts provide at least suggestions for the reclassification of the *Hyracoidae*. Apart from the *Cervidae* and *Hyracoidae*—where the presence of the external *fabella* may throw light on their evolutionary history—we are forced to the conclusion that the large (and mixed!) order of the ungulates presents no sesamoids of the heads of *gastrocnemius* or *popliteus*. The absence of the *fabellae* in ungulates has been frequently noticed§, but so far we believe no stress laid on its evolutionary importance.

* We failed to find any sesamoid of the knee-joint on dissecting a young *Tragulus*: see below.

† On the fossil *Megaceros hibernicus* (Irish deer) we were unable to find any trace of a *fabella*.

‡ *Proc. Zool. Society*, 1865, p. 350.

§ Meckel, Gruber, etc. See also *Journal of Anatomy and Physiology*, Vol. XXXII. p. 750, 1898.

(11) *Pinnipedia.*

No sesamoids of the *gastrocnemius* were found by Gruber* in *Phoca* sp.? He thus confirmed an earlier statement of Meckel† that the *fabellae* were absent in seals. Again reminding the reader of the unreliability of evidence drawn from articulated skeletons, we may note that the following articulated skeletons were examined without finding traces of *fabellae*: *Otaria stelleri* (Steller's sea lion), *Stenorhynchus serridens* (saw-toothed seal), *Cystophora cristata* (bladder-nosed seal), *Otaria falklandina* (Falkland fur seal), *Phoca vitulina* (common seal), *Phoca greenlandica* (the harp seal), *Phoca hispida* (ringed seal), *Balaenoptera rostrata* (lesser fin whale), *Odoboenus rosmarus* (walrus), *Macrorhinus leoninus* (elephant seal), *Otaria jubata* (southern sea lion).

In *Otaria californiana* there is small process or upward growth where articular surface of external condyle meets popliteal area, but it is not sesamoidal in character. We dissected, however, a knee-joint of *Otaria australis* (South American fur seal) and found no such process. There is nothing, therefore, in our observations inconsistent with the view of Meckel that the *Pinnipedia* have no *fabellae.*

(12) *Chiroptera.*

Meckel‡ asserts that the bats have only the external *fabella.* Owen§ says that *Vespertilio murinus, Pteropus,* and *Galeopithecus* have both external and internal *fabellae,* while Gruber says that *Galeopithecus* has neither *fabella.* Blainville‖ asserts the existence of a sesamoid in the tendon of *popliteus* in the case of *Vespertilio L.,* i.e., of a *cyamella.* Humphry¶ dealing with the myology of limbs of *Pteropus* and treating of *gastrocnemius* writes: "There is a minute ossicle in the outer head. I cannot find one in the inner head." The not unusual statement is that the sesamoids, the patella and even the semi-lunars are wanting. The above statements suggest that *fabellae* and *cyamella* exist occasionally or in minute forms. Parsons** writing of the *Chiroptera* says no movement except extension and flexion is allowed in fruit bats and there are no traces of the semilunar cartilages; the patella is absent. "In the long-eared bat (*Plecotus*) which will serve as an example of the insectivorous bats the knee-joint does allow a certain amount of rotation and in it semilunar cartilages are found as very delicate rings. This bat resembles the fruit bat, however, in the absence of any trace of a patella."

On the other hand the bat *Cynopterus marginatus* appears in the specimen (R. C. of S. Museum A. 3396) we were able to examine to have an ossified patella, and there might be other ossifications in the knee-joint, but it was not possible to say from this specimen. Gruber†† also found the external *fabella* in *Phyllostoma hastatum,* but failed to find it in *Pteropus* sp.? Besides the above *Cynopterus* we have obtained and examined a considerable number of knee-joints of bats—principally

* *Loc. cit.* p. 60.
† *Loc. cit.* S. 634.
‡ *Loc. cit.* S. 634.
§ *Anatomy of Vertebrates,* Vol. II. p. 358.
‖ *Ostéographie des Cheiroptères,* Tom. I. Fasc. 5, p. 30, 1840.
¶ *Journal of Anatomy and Physiology,* 1869, p. 313.
** *Ibid.* Vol. XXXIV. p. 310, 1900.
†† *Loc. cit.* p. 55.

Pteropus—having the muscles attached, but we have failed to find any sesamoids. It is probable that a microscopic examination of sections from moist material would lead to more conclusive results, and, perhaps, indicate minute patches of hyaline cartilage hardly appreciable by ordinary dissection. For our present purposes, however, such an investigation would be very unlikely to change the significance which can be drawn from the results for *Chiroptera*. In this order the sesamoids of the knee-joint appear occasionally, as in the higher primates, or in minute forms and are vestiges of a complete system of patella, *fabellae*, and *cyamella*, which belonged to their evolutionary ancestors. Even in this case which is merely vestigial, it is worth while noting the greater importance of the external *fabella*.

(13) *Insectivores.*

With this order we reach vestiges of a more primitive state of affairs in the sesamoids of the knee-joint. The *lunulae* common in the Rodents and carrying us back to reptilean forms are frequently preserved. In the carnivores and the higher primates they occur only as anomalies or have disappeared entirely. According to Blainville* *all* insectivores have both *fabellae*. Owen† qualified this by saying that *most* insectivores have both. Meckel‡ had already remarked that the mole has both *fabellae*, but the hedgehog he reported as having only the external *fabella*. Gruber§ on the contrary found the mole to have an external *fabella* only. He reports that both *fabellae* are wanting in *Erinaceus europaeus* and *E. auritus*, and again in the shrew mouse, both *Sorex vulgaris* and *S. fodius*. He examined three cases of *Myogale moschata*, the Russian desman. In the first case there were no *fabellae*; in the other two cases no internal but external *fabellae*.

In den Fällen aber mit Vorkommen des Ossiculum externum sass dieses in einem Ausschnitte am Ende des scharfen Angulus externus über dem Condylus externus und daneben in einer kleiner Grube an der hinteren Fläche der Femur‖.

In the case with no *fabellae* there was a "platter Fortsatz" on the same spot. Gruber found this also in *Myogale pyrenaica* where there were no ossicles. We dissected two moles, *Talpa europaeus*, and found the external *fabella* only; thus our specimens agreed with Gruber's and not Meckel's. The hedgehog (*E. europaeus*) examined by us had both *fabellae* and both lateral and mesial anterior *lunulae* (see Plate XXIII, Figs. 56*a* and 56*b*). Thus Gruber's statement is not universal. In a specimen of *Myogale moschata* we found only a large external *fabella* of unusual shape and size, and in *Myogale pyrenaica* no *fabellae*; these two latter in mounted specimens only. In mounted specimens we found only a large external *fabella* in *Talpa europaeus*, *Macroscelides* (elephant shrew), *Solenodon cubanus* and *Tupaia javanica*. In a specimen of *Tupaia picta* in our laboratory both *fabellae* are present, and both occur in *Tupaia tana*, the Borneo tree shrew. We have also found both *fabellae* in

* *Ostéographie des mammifères*, Tom. I. Fasc. 4, p. 47, Paris, 1841.

† *Anatomy of Vertebrates*, Vol. II. p. 393.

‡ *Loc. cit.* S. 634. § *Loc. cit.* p. 56.

‖ Gruber notices (*loc. cit.* p. 64, that the external *fabella* in *Myogale* is sometimes fused to femur in a special *Fortsätzchen*. This is the only reported case of ankylosis of a sesamoid we have come across, except in the Marsupialia.

Hylomys suillus dorsalis (see Plate XXIV, Figs. 62 *a* and 62 *b*) and *Rhynchocyon cerni* (long-nosed jumping shrew). Two mounted specimens of the golden mole (*Chrysochloris aurea*) provided no signs of *fabellae*. A mounted specimen of *Potamogale* provided an external *fabella*, and a dissection of *P. velox* confirmed this. In *Ericulus setosus* we found no *fabellae*, but the *cyamella* and two anterior *lunulae*, an unusual combination.

Plate XXIV, Fig. 59, provides a drawing of the lateral aspect of the left knee-joint of *Potamogale velox*. There was no sign of ossification in the popliteus tendon and only the external *fabella* was present. Fibula and tibia are fused at their distal ends.

In *Hylomys suillus dorsalis* (Plate XXIV, Figs. 62 *a* and 62 *b*) both *fabellae* are present and fibula and tibia are again distally fused. There was no *cyamella*, and no *lunulae* could be found.

In the mole only the external *fabella* was present and there was no sign of ossification in the popliteal tendon. No *lunulae* could be discovered, but in such small animals absolute certainty could only be reached by cutting a series of sections of the semilunars.

Parsons* in his paper on the limb myology of *Gymnura rafflesii* notes the presence of an orthosesamoid in the external head of *gastrocnemius*; he does not refer to the existence of *lunulae* or *cyamella*. To sum up, the more primitive forms of the *Insectivora* probably possessed both *fabellae*, a *cyamella*, and mesial and lateral anterior *lunulae*. The external *fabella* has been generally preserved, but the complete system as illustrated in the hedgehog shows signs of disappearance by the loss of one or more members as we also find in the higher Primates or Carnivores. It is, however, very far from being vestigial as in the case of the *Chiroptera*.

(14) *Rodents.*

According to Meckel the marmot, the common hamster, the guinea-pig, the agouti, the jerboa (*Mus sagitta*) and the hare have both *fabellae*, but the beaver, the squirrel, the rat, the dormouse and the Cape mouse (*Aorychus capensis*) only the external *fabella*†. Bartolinus‡ had before Meckel referred to the two *fabellae* in the hare and Tyson to those in the guinea-pig. Owen§ found both *fabellae* in the squirrel as well as a *lunula*. He also found both *fabellae* in *Mus rattus*, the water vole (*Arvicola amphibia*), *Hydromys chrysogaster* (golden-bellied rat) and *Hystrix cristata* (common porcupine). Gruber made a fairly long investigation as to the *fabellae* in rodents. He found both *fabellae* in the following: *Myoxus murinus* (the fat dormouse), *Sciurus vulgaris* (common squirrel), external larger; *Pteromys volans* (? American flying squirrel), external larger; *Tamias striatus* (= *Tamias asiaticus*?); *Spermophilus citillus* (European souslik); *Mus musculus* (common mouse); *Mus rattus* (common rat), external larger; *Hypudaeus* (? *Mus*

* *Journal of Anatomy and Physiology*, 1898, p. 322.

† *Loc. cit.* S. 604—5.

‡ See Part I of this memoir, p. 147.

§ *Anatomy of Vertebrates*, p. 384.

amphibius, the water vole); *Lemmus obensis* (Siberian, ? Norwegian, lemming), internal somewhat larger! *Fiber zibethicus* (musquash), external large and internal small; *Dipus jaculus* (? *Alactaga jaculus*); *Spalax typhlus* (mole rat); *Lepus timidus** (hare), external in facet on external condyle, internal in groove, but this may happen with external; *Lepus cuniculus* (rabbit), internal small as compared with external; *Cavia* (guinea-pig), external larger; *Dasyprocta aguti*, both as hemi-sesamoids† (hyaline cartilages). On the other hand in a *young* specimen of *Castor fiber*, he found neither and this was also the case in *Cercolabes prehensilis* (tree porcupine), although *gastrocnemius* was very fully developed. In *M. glis* Gruber found the external *fabella* only, articulating on condyle. He examined three kinds of *Cricetus* (hamster) and found in one both *fabellae*, in a second the external only, and in a third neither.

We now come to our own investigations. In the authorities cited above we have no records of the presence of the *cyamella* and only the single reference of Owen to the presence of a *lunula* in the squirrel.

A dissection of the common rabbit showed both *fabellae* and the *cyamella* but no *lunula* (see Plate XXIV, Fig. 58). In a mounted skeleton, however, we found both *fabellae*, the *cyamella* and both anterior *lunulae*. A dissection of the common squirrel (two specimens) showed both *fabellae*, the *cyamella* and an anterior mesial *lunula* (see Plate XXIII, Fig. 55 *b*). In a third very young squirrel the lateral *fabella* was found, but no mesial had yet developed (no microscopic examination made); the *lunula* was well developed‡. In the African squirrel (*Paraxerus jacksoni*) we found both *fabellae*, the *cyamella* and the anterior mesial *lunula* (see Plate XXIV, Figs. 60 *a* and 60 *b*).

In the domestic mouse both *fabellae* and both anterior *lunulae* but no *cyamella* were found; the smallness, however, of the knee-joint renders (without microscopic examination of sections) the discovery of the sesamoids rather precarious.

In the musk rat (*Fiber zibethicus*) we found both *fabellae*, two anterior *lunulae*, of which the mesial is under the ridge of the patellar depression, and one posterior lateral *lunula*. There was no trace of a *cyamella*, and it is possible that it does not exist in the musk rat or is replaced by this posterior *lunula* (see Plate XXIV, Figs. 61 *a* and 61 *b*).

Turning to mounted specimens we found both *fabellae* in the following: *Trichys guentherai* (Gunther's porcupine), *Lagostomus trichodactylus* (Viscacha§) in two specimens, *Thrynomys swindernianus* (cave rat), *Mus fuscipes* (brown-footed rat), *Mesembriomys hirsutus* (Gould's jerboa rat), *Microtus amphibius* (water vole), *Alactaga decumana* (Kirchig jerboa), *Alactaga acontium* (Siberian jerboa), *Spermophilus guttatus* (a souslik), *Sciuropterus volucella* (North American flying

* Mivart and Murie, *Proc. Zool. Society*, 1866, p. 413, state that the hare differs from the guinea-pig by having neither *fabella* (!).

† We have found orthosesamoids in our specimen.

‡ We also found only the lateral *fabella* in a mounted specimen of the common squirrel, but this is hardly evidence even of the occasional absence of the mesial. Cf. Meckel's statement p. 357.

§ There is a double-sized lateral and a smaller mesial *fabella*.

squirrel), *Dasyprocta aguti* (golden agouti)*, *Cavia porcellus* (guinea-pig), *Myopotamus coypus* (coypu), in a second specimen only the external *fabella* was preserved, *Anomalurus fraseri*, in this case the *cyamella* had been preserved.

In a skeleton of *Dolichotis patachonica* (Patagonian cavy) no *fabellae* remained, but there were exceedingly well-marked depressions for the *fabellae* on the condyles†, and in another skeleton of *Dasyprocta aguti* where the two *fabellae* had disappeared their positions were well marked by depressions on the condyles. There were well developed pits also on the condyles, almost certainly for *fabellae*, in *Hystrix leucura*. Occasionally when the *fabellae* were not traceable, *lunulae* had been preserved; thus in *Hydromys chrysogaster* (golden-bellied rat) we found both anterior *lunulae*. In another specimen, however, we found both *fabellae* and the anterior *lunulae in situ*. These are reproduced in Plate XXIII, Fig. 57 *a*. The *lunulae* of the anterior border of the semilunar cartilages existed in another skeleton of *Dolichotis patachonica*†. In a mounted specimen of *Hystrix cristata* (crested porcupine) the *fabellae* and *cyamella* failed but there appeared to be an external anterior *lunula*, while in *Synetheres insidiosus* (a South American porcupine) there were, if there ever had been any, no remaining sesamoids, not even the *lunulae*.

Parsons, who has dealt at length with the myology of the rodents, wrote as follows in 1894‡ of the *Hystriomorphinae*:

> The presence or absence of the *fabellae* does not seem to depend on the affinities of the animal, as they are large in *Aulacodus* on both sides, while in *Myopotamus*, only the outer one is present. In *Dasyprocta* they are both present, in *Calogenys* both absent.

We have found in a specimen of *Myopotamus* both *fabellae* present.

Again:

> In the *Sciuromorphinae*, however, they were found in every case except that of *Castor canadiensis* (*Sciurus*, *Pteromys*, *Xerus*, *Spermophilus*, *Arctomys*, *Castor*).

In a paper of 1896§ on the "Myology of the Rodents," Parsons writes that all except fam. *Spalacidae* (*Rhizomys*, *Georychus*, *Bathyergus*) have two *fabellae*.

It will be seen that a good deal of work remains to be done on the rodents. It is not easy to obtain adequate material to determine an isolated problem of the knee-joint like the present. It is to be hoped that myologists in the future may pay attention to the existence of *cyamella* and *lunulae* as well as of the *fabellae*. But our experience shows us that without a most careful examination, which in many cases ought to be microscopic, it is by no means easy to ascertain the presence or absence of these sesamoids even in most specimens of the smaller rodents. Further, it is quite clear from the material we have gathered together that persistent presence or absence in the same species cannot be asserted from the examination of isolated specimens.

* Windle found both *fabellae* in *Dasyprocta isthmica*, *Journal of Anatomy and Physiology*, 1896—97, p. 352, and Mivart and Murie, *Proc. Zool. Society*, 1866, found both in *Dasyprocta cristata*.

† Windle, *Journal of Anatomy and Physiology*, 1896—1897, p. 352, says he found only the lateral *fabella*.

‡ "Myology of the *Sciuromorphinae* and *Hystriomorphinae*," *Proc. Zool. Society*, 1894, p. 291.

§ *Proc. Zool. Society*, 1896, p. 181.

The *Rodentia* undoubtedly show, however, a greater persistency of the *fabellae* than the *Insectivora*, and what is more a greater prevalence of *cyamella* and anterior *lunulae*. Had these been directly searched for by others as well as ourselves, we believe their prevalence would have been much more emphasised in the reader's mind. The appearance in a few cases of the external posterior *lunula* is also of much suggestiveness. We have not, however, yet come across a case of this *lunula* co-existing with the *cyamella**. In the rodents we have further evidence of the difference in origin of the two *fabellae*, partly in the fact that when both are present the lateral is larger and more important than the mesial, it being as in man an attachment of more than one muscle, and partly in the second fact that, in those species which have a single *fabella* only, it is invariably the mesial which fails.

Before we pass to the *Edentates* as linking up the groups we have already discussed we must consider the Primates themselves and ascertain whether they obey the same general law, namely that the lower and more primitive forms are multi-sesamoidal with regard to their knee-joints, while the higher types have retained fewer, or if they exhibit any at all it is only as anomalies.

(15) *Primates.*

We have already discussed Man. The lateral *fabella* occurs as an anomaly and may be either a hemisesamoid or an orthosesamoid. The mesial *fabella* as an orthosesamoid has not yet been satisfactorily demonstrated†. It is clearly far rarer than the lateral *fabella*. There is evidence for the *cyamella* as a very rare anomaly and possibly still more rarely of *lunulae*. The comparative frequency of these anomalous sesamoids of the knee-joint in men of different races would be a study of considerable interest and might give valuable hints. It should not be hard by means of skiagrams to obtain something like comparative figures for, say, Japanese, Negroes‡, Australians and American Indians which might be set against our English work.

Turning to the *Anthropoidea* we meet the difficulty so often referred to, namely, that the larger skeletons are apt to be too much cleaned for present purposes.

* The reader will have noted that we have dropped the adjective "lateral" in this latter portion of our work. So far we have no evidence beyond Pfitzner's for cats and our own still more doubtful for man of the existence of the mesial *cyamella* even as an anomaly.

† In 1909 Lunghetti asserted that he had found the mesial *fabella* in two cases, but he does not give evidence that they were orthosesamoidal, *Internationale Monatschrift für Anatomie und Physiologie*, Bd. XXVI. S. 71. L. Stieda in 1902 found and exhibited to the German Anatomical Society (*Anatomischer Anzeiger. Verhandlungen der Anatomischen Gesellschaft. Ergänzungsheft zum Bd.* XXI. 1902, S. 127) a mesial *fabella* from a human cadaver. It articulated with the mesial condyle, and was enclosed in the knee capsule. It could be clearly seen and felt. Stieda says it was 1 cm. about in diameter, and therefore a very large size for even a lateral *fabella*. He did not apparently extract the *fabella* and so make himself absolutely certain that it was ossified. He had possibly not fully studied the difficulties of the subject for he describes (S. 128) Heister's figure as "eine sehr gute und lehrreiche Abbildung"! See our p. 150 and Plate II.

‡ Chudzinski according to Poirier holds that the sesamoid bone of the *inner* head of *gastrocnemius* is invariable in the Negro! We have been unable to verify the reference and very strongly doubt the truth of the assertion. Dr E. C. Derry kindly examined for us two negro cadavers in his dissecting room and found in both cases neither mesial nor lateral *fabellae*.

Meckel does not seem to have examined the *Anthropoidea* although he deals fairly fully with the lesser apes. Gruber has also no data. Owen's paper of 1866 contains no reference as to the sesamoids of the knee-joint* Macalister in his paper on "The Muscular Anatomy of the Gorilla"† writes that

> The popliteal muscle had a sesamoid bone or cartilage; it [the muscle?] was about as large proportionately as in man.

Thus Macalister seems the first to have noted the *cyamella* in the gorilla. We have already drawn attention to the fact that Camper had found it in the orang in 1791. Macalister in a further paper‡ dealing with the chimpanzee, remarks:

> The popliteus was small and had no sesamoid nodule in its tendon thereby agreeing with Wilder's and differing from Vrolik's specimen. Trail did not find it [*? popliteus*] present.

Wilder in his paper on "The Comparative Myology of the Chimpanzee"§ states, when dealing with *popliteus*, that he "could not find the cartilaginous nodule in the external lateral ligament where the muscle arises." It is not clear that this really does refer to the *cyamella* Wilder continues: "Trail and Tyson could not find the muscle, but it was present in Vrolik's chimpanzee and in the gorillas of Wyman and Duvernoy" (*l.c.* p. 375). The "it" of Macalister's sentence just cited should refer to *popliteus*, otherwise Macalister has misread Wilder, who is referring to the muscle and not to the sesamoid‖.

It would seem then that of the anthropoids the gorilla, chimpanzee and orang have no *fabellae* and that gorilla and orang have the *cyamella*. The chimpanzee on the weight of the evidence published above does not seem to have the *cyamella*. Reports that the chimpanzee has it arise from a not unnatural reading of Macalister's loose wording, at most it can only be as an anomaly. Thus even in this respect man approaches more closely to the chimpanzee than to the gorilla or orang.

By the courtesy of Sir Arthur Keith we were able to dissect the knee-joint of a chimpanzee. We found no *fabellae*, and not a *cyamella*, but a scarcely perceptible thickening of the popliteal tendon¶. We procured and dissected the knee-joints of two orangs. In neither case was either *fabella* present but in both cases there was a large *cyamella*: see Plate XXVI, Figs. 68 and 69. Sir Arthur Keith kindly allowed one of us to examine his manuscript thesis of 1894 entitled: "The Myology of the Catarrhini. A Study in Evolution." It is a most valuable work of 600 pages presented to the Library of the Royal College of Surgeons in 1919. In this work he notes that he has not found the *fabellae* in gorilla or orang, but he found the *cyamella* (described as sesamoid in the tendon of *popliteus* above

* "Osteological Contributions to the Natural History of the Anthropoidal Apes, No. VII." *Trans. Zool. Soc. London*, Vol. v. The femora of gorilla, chimpanzee and the orang are discussed on p. 14, *et seq.*

† *Proc. Royal Irish Acad.*, Vol. I. 2nd series, p. 505. Dublin, 1870, p. 74.

‡ "Myology of the Chimpanzee and other Primates," *Annals of Natural History*, 1871, p. 9.

§ *Boston Journal of Natural History*, Vol. VII. p. 375, 1861.

‖ Gruber says Vrolik did *not* find the sesamoid; and we think Gruber naturally read Macalister's sentence to indicate that Vrolik had found the sesamoid and he wished to contradict Macalister.

¶ Our chimpanzee was like those of Wilder, Macalister and Vrolik in the fact that *popliteus* was present.

external tibial tuberosity) in both, but not in the chimpanzee (p. 307). On the other hand he states that he did find in a chimpanzee (p. 297) a mesial *fabella*—i.e. the inner tendon of head of *gastrocnemius* contained a sesamoid. This result is of considerable importance. The consensus of opinion is against the occurrence of any *fabellae* in the chimpanzee but the occasional anomalous appearance of a *fabella*, especially a mesial hemisesamoid, would be exactly what we might anticipate with the close relationship that exists between man and the chimpanzee.

Hylobatinae. Hepburn in his memoir on "The Muscles and Nerves of the Anthropoid Apes" (*Journal of Anatomy and Physiology*, 1892, p. 337), found a sesamoid bone developed in each tendon of origin; in dealing with *popliteus* he makes no mention of the existence of a sesamoid, which he would have been pretty certain to have done had it existed. We dissected the knee-joint of a gibbon (*Hylobates muelleri*) and found the mesial *fabella* only, no *cyamella** and no *lunulae*: see Plate XXVII, Fig. 72. This result is in full accord with Sir Arthur Keith's expressed in his thesis: "Gibbon: sesamoids extremely variable even in adult animals. There may be one in each head or both may be absent or either may be absent" (p. 297). In Diagram 75 of Sir Arthur Keith's thesis figures the knee-joint of a gibbon, the internal head of *gastrocnemius* with large, the external head with minute sesamoid. He has never observed the *cyamella* in the gibbon. The existence of individual gibbons in which the mesial *fabella* only appears, while it may seem like the exception which proves the rule, is not so really, for the rule is that no species occurs in which the mesial *fabella* is invariably present and the lateral *fabella* invariably absent. The invariable presence of the lateral and invariable absence of the mesial *fabella* is as we have seen characteristic of certain species†.

The mounted skeletons of *Anthropoidea*, e.g. those at the British Museum (Natural History) and at the Royal College of Surgeons, are all "too clean" to show the sesamoids of the knee-joint. It may be hoped that in future the *cyamella*, where it occurs, will be preserved and mounted as well as the *fabellae*, both in these cases and those of the Old and New World Monkeys, although very often the *fabellae* are preserved in the latter.

Simiadae (Old World Monkeys). We have already seen that the existence of both *fabellae* in the Old World Apes was known to Sylvius and Riolanus (see our pp. 144—145). Meckel gives a fairly extensive list of lemuroids, New World and Old World Monkeys in which he had observed both *fabellae*‡. Gruber notes that he had found both in the genera *Cercopithecus, Inuus, Cynocephalus*§, and *Cebus*

* The tendon of *popliteus* was "horny," but there was no sesamoid.

† It is worth noting that even in the gibbon Keith (*Thesis*, p. 298) found the lateral *fabella* more frequently present than the mesial.

‡ *Loc. cit.* S. 634—635.

§ We have found both *fabellae* in *Macacus leoninus*, *M. pileatus*, *M. rhesus* (several specimens), *M. inuus* (Barbary ape), *M. cynomolgus* (several specimens), *Cynocephalus anubis*, *M. nemestrinus* (two specimens), *M. speciosus* (Japanese ape), *Cercopithecus lalandii* (two specimens), *Colobus ursinus*, *C. vellerosus*, *Nasalis larvatus* (two specimens), *Semnopithecus orientalis*, *S. entellus*, *Papio hamadryas*, etc. etc. Burdach, Olg and others have noticed other cases from 1838 onwards.

(New World Monkey). In *Cynocephalus* he found sometimes the mesial sometimes the lateral the larger. In the remainder the lateral was invariably the larger. Neither Meckel nor Gruber* refers to the *cyamella* as existing in either *Simiadae* or *Cebidae*.

Keith in the above-mentioned thesis takes *Semnopithecus* as his type. He thus describes the relation of the muscles in his type to the sesamoids:

Gastrocnemius. Origin. *External head*; arises by musculo-tendinous fibres from a sesamoid embedded in the general capsule of the knee-joint and plying over the upper convexity of the external femoral condyle, and from the shaft of the femur immediately above and external to the femoral condyle (p. 296).

Internal Head: from a sesamoid situated similarly to that of the external head and from the shaft of the femur immediately above the internal condyle.

Plantaris. Origin; from the sesamoid in the external head of *gastrocnemius* (p. 304).

Popliteus. Origin. By a rounded tendon from the lateral aspect of the external femoral condyle beneath the external lateral ligament and within the capsule of the knee-joint....There is a sesamoid in the tendon over the external tibial tuberosity (p. 307).

The arrangement of the muscles relative to the sesamoids in *Semnopithecus* as given by Keith is curiously like what we have learnt of the relation of the muscles to the *fabellae* and *cyamella* when they occur in man. This is peculiarly true of *plantaris*, and is evidence, similar to that provided in the memoir on the Femur by Pearson and Bell, that in a number of respects man is closer to the *Simiadae* than to the *Anthropoidea* whether by retaining more primitive characters, or by reverting to earlier types as better suiting his differentiated development.

The *cyamella* as well as both *fabellae* are present not only in *Semnopithecidae*, but in macaques, *Cynocephali* and practically all the *Simiadae*, we have been able to investigate†. Our drawing (Plate XXVII, Figs. 71 *a* and 71 *b*) shows the two *fabellae* and the *cyamella* in *Mycetis palliatus*, both in position and after extraction.

Lastly we note that we have not been able to find *lunulae* in any knee-joint we have examined of the *Simiadae*. Should they occur, we believe their appearance is very rare.

Cebidae (New World Monkeys). Here the problem is somewhat easier than in the *Simiadae*. The bulk of the species being smaller in size our purchased specimens were less "clean," i.e. the muscular attachments remained. We were thus better able to give positive evidence as to the non-existence of *lunulae*.

(i) We found both *fabellae* in *Hapale aurita* (white-eared marmoset), *Midas oedipus*, *M. midas*, *M. ursulus* (two specimens), *M. auritus* (golden marmoset),

* It is the more noteworthy in Gruber's case for he does refer to a large *cyamella* in *Stenops tardigradus*.

† We have noted *cyamella* in *Nasalis larvatus*, in *Macacus* (the Bonnet monkey), *M. inuus*, etc. and believe that it is general in the *Simiadae*, although usually cleaned off in mounted specimens.

Hapale jacchus (common marmoset), *Jacchus auritus*, and two other specimens (sp. ?) of *Jacchus*, *Cebus capuchinus* (two specimens), *Cebus* (sp. ?) (two specimens), *Lagothrix humboldtii* (two specimens), *Pithecia satanas* (Black Sahi), *Pithecia* (sp. ?), *Nyctipithecus*, *Chrysothrix sciurea* (three specimens).

(ii) In none of these were any *lunulae* to be found.

(iii) We found the *cyamella* definitely in *Hapale jacchus*, *Jacchus auritus*, in two other specimens of *Jacchus* (sp. ?), in *Midas ursulus* (two specimens), *M. midas*, *M. auritus*.

We could not definitely assert its absence in *Cebus* or *Pithecia*, or the squirrel monkeys, as the material was less complete.

The case of *Ateles* is also doubtful; two specimens were examined, in neither was a *cyamella* found; one possessed no *fabellae* and the other only an internal one. This may be due to rough usage, or it might emphasise the correspondence with the gibbons which Pearson and Bell noticed in considering the femur of *Ateles*.

While we should have anticipated finding *lunulae* in the more primitive types of *Cebidae*, we have not done so, and conclude that the *lunulae* were lost to the Primates except as occasional anomalies before the separation of *Simiadae* and *Cebidae*. The chief difference between the apes of the New and Old Worlds is that the *Cebidae* (except in the possible case of *Ateles*) have never progressed to the stage of the *Hylobatinae* with loss of *cyamella* and to the stage of the *Anthropoidea* with loss of *fabellae* as well.

Lemuroidea. We naturally turn with much interest to the prosimian group; for the sesamoids of the knee-joint in their case ought to link up with those of the Rodents and Insectivores and the test of this will clearly be the occasional or persistent occurrence of *lunulae* in one or more species.

Here again the expense of procuring moist material, especially when it is to be investigated only for a single point in as wide a range of species as possible, has much handicapped our work*.

We turn first to the lemurs proper and notice the following remarkable results†: Both *fabellae*, the *cyamella* and the lateral anterior *lunula* were found in *Lemur varia*, *L. catta*, *L. melanocephalus*, *L. coronatus*, *L. macaco leucomystax*‡. There is no doubt therefore that the *Prosimia* add the lateral anterior *lunula* to the *fabellae* and *cyamella* of the *Simiadae*. See Plate XXVII, Figs. 70 and 73.

Turning to *Galago* we have found: *Galago alleni*, both *fabellae*, the external very large, the *cyamella* and the lateral anterior *lunula*. *G. crassicaudata* ex-

* The aid given by the Government Grant Committee to the Study of Pearson and Bell on the femur, for the purchase of primate material, has of course aided us.

† Meckel notes merely the existence of both *fabellae* in *L. mongoz*, and *L. albifrons*. Gruber states that *Stenops tardigradus* lacks both *fabellae* but has a large articulating sesamoid in the tendon of *M. popliteus*.

‡ Femora without the semilunars of *L. rubiventer*, *L. niger*, and *L. macao varius* showed both *fabellae* and in the first case the *cyamella* also remained.

hibited also the two *fabellae**, *cyamella* and lateral anterior *lunula*. Another *Galago* (sp. ?) gave the same result, while a series of mounted specimens provided sometimes (as oversights of the articulator !) the *fabellae*, or the *cyamella* and once a *lunula*.

In *Lemur macao niger*, both *fabellae*, the *cyamella* and apparently *both* anterior *lunulae* were present. *Nycticebus javanicus* had also *fabellae*, *cyamella* and lateral anterior *lunula*; a mounted specimen of *Nycticebus tardigradus* retained only the *cyamella*†. Going further we find *Loris gracilis* with *fabellae*, *cyamella* and lateral anterior *lunula*, apparently but difficult to decide definitely without a section there was also a lateral posterior *lunula*; a mounted specimen showed more definitely two *lunulae*.

Perodicticus potto gave us *fabellae*, *cyamella* and lateral anterior *lunula*, and the *cyamella* was confirmed in a mounted specimen of the Calabar potto.

We now reach cases in which we can be more or less definite as to absence of *lunulae* because the semilunars were preserved.

Chiromys (Aye-Aye) dissected gave both *fabellae* and *cyamella*, there were no *lunulae*: see Plate XXVIII, Fig. 74 *a*. A second specimen in the Biometric Laboratory was investigated with precisely the same result no *lunulae*. Mounted specimens show the *fabellae* as a rule; in one we found the *cyamella* preserved. The existence of the *cyamella* in the Aye-Aye was, we believe, first noted by Owen. He makes the important point that "there is a sesamoid in the external lateral ligament of the knee-joint at its insertion into the head of the fibula." (See our Plate XXVIII, Fig. 74 *b*.) He also refers to the *fabellae* in the Aye-Aye. ("On the Aye-Aye." *Trans. Zool. Soc. London*, Vol. v. p. 53, 1866.)

In *Avahi laniger* and *Hapalemur griseus* we found both *fabellae*, definitely no *lunulae*, the *cyamella* failed but might possibly have been removed. A mounted specimen of *Avahi laniger* as well as one of *Propithecus diadema* showed only the *fabellae*. In a mounted specimen of *Lepidolemur microdon* we found only *cyamella* and external *fabella*.

Chirogaleus and *Indris brevicaudata* gave only the *fabellae* and *cyamella*.

Lastly *Tarsius spectrum* has both *fabellae* and the *cyamella*. Curiously enough as in *Galago crassicaudata* the mesial *fabella* may be larger than the lateral, although we have seen specimens mounted having solely the lateral *fabella*. We can only say we have not yet found *lunulae* in *Tarsius*. This is not very strong evidence against their existence, because the semilunars are so small that without fresh moist material carefully sectioned and microscopically examined it would be difficult to give a strong negative. When we resume our results for the *Lemuroidea*

* Mesial larger than lateral.

† Mivart and Murie (*Proc. Zool. Soc. London*, 1865, p. 129) in their "Anatomy of *Nycticebus tardigradus*" note of *M. popliteus* that it is inserted by a very strong tendon containing a large sesamoid bone...into the depression outside the lateral condyle. They do not mention when discussing *M. gastrocnemius* the existence of sesamoids therein.

we see their great importance. The non-appearance of the *lunulae* in the marmosets is more or less accounted for, because in the lower stage of the lemuroids while they are found, they are not universal; the *lunulae* have already begun to disappear, although the type lemuroid may be said to be that which has both *fabellae*, the *cyamella* and the lateral anterior *lunula*. A further point which will develop its interest as we proceed is that noted by Owen in the Aye-Aye, namely the situation of the *cyamella*. We reproduce, Plate XXVIII, Fig. 74 *b*, Owen's drawing. The reader will note how in a marvellous manner the *cyamella* and the lateral *fabella* are being brought into close association. The bearing of this on the origin of both of them will be seen in the sequel. The *cyamella* articulates with the head of the fibula and the possibility of the lateral *fabella* articulating with the *cyamella* is seen to exist*. Fig. 74 *a* shows our actual dissection; the head of the *fibula h* is in contact with the *cyamella f*. The two branches of *gastrocnemius a* and *b* have been deflected to show the *fabellae c* and *d*, as dark patches; the positions in which they rest on the condyles are also indicated.

Plate XXVIII, Fig. 77 gives the dissection of the knee-joint of *Indris brevicaudata* and shows the *fabellae* and *cyamella* in position.

Plate XXVII, Figs. 70 *a* and 70 *b* show in *Lemur varia* the type knee-joint of the lemurs. *a* is the external, *b* the internal *fabella*, *c* the *cyamella* on the head of the fibula and *d* is the lateral anterior *lunula*. Figs. 75 and 73 show with the same notation the same four sesamoids in *L. catta* and *L. melanocephalus*.

Plate XXVIII, Fig. 76 gives the lateral aspect of the dried knee-joint of *Perodicticus potto*. In this case no *fabellae* were present, having most probably been knocked off. The *cyamella* (*cy.*) is seen resting on the head of the fibula, the popliteal tendon sends a branch down to the head of the fibula. On the posterior and upper face the *cyamella* is concave, and this hollow seems designed for the *fabella* to rest in. The anterior external *lunula* is in its usual position, i.e. in the semi-lunar cartilage just in front of the popliteal tendon.

The main point of interest in the lemuroids is undoubtedly the link that the *lunula* presents with the Rodents and Insectivores. But the close relationship which the forms and positions of *cyamella* and lateral *fabella* in the lemuroids bear to a continuous articulating system connecting the head of the fibula with the lateral condyle of the femur is full of suggestiveness. This relationship of lateral *fabella* and *cyamella* explains we think to some extent Gruber's confusion of the two when as in man only one occurs and then as an anomaly.

(16) *Edentates.*

Before we discuss the very peculiar sesamoidal knee-joint characters of this order it seems desirable to be clear on a point which is rather obscure in some of the earlier writers, who are liable to confuse the *cyamella* with the lateral *fabella*.

* It is worth while comparing the drawing of the cat's knee-joint on our p. 352 with Owen's of the Aye-Aye's knee-joint. The manner in which with flexed limb the lateral *fabella* and *cyamella* come in contact will be obvious.

The following rules appear to us to hold for the somewhat heterogeneous material classed as *Edentata*.

(i) *Lunulae* may occur.

(ii) There are no cases at all of the mesial *fabella* (with or without the lateral *fabella*) occurring*.

(iii) When the *cyamella* occurs it is very extensive and when it occurs there is no lateral *fabella*.

(iv) Conversely when the lateral *fabella* occurs there is no *cyamella*.

(v) There are cases in which no sesamoids whatever of the knee-joint have been found.

We have accordingly reached a stage in which the mesial *fabella* has not been evolved, for it does not seem to exist in any earlier types. Further we have reached a stage where a non-sesamoidal type of *Monodelphia*—a type afterwards so widely developed in the *Ungulata*—is seen alongside a sesamoidal type afterwards so widely developed in *Rodentia, Insectivora, Primata* and *Carnivora*. It would seem as if the *Edentata* as modern representatives of these primitive *Monodelphia* must be from the knee-joint sesamoidal standpoint most suggestive from their very heterogeneity. Meckel speaks of the external *fabella* only as existing in the Edentates, and cites as examples the Anteater and Sloth (Aï). He does, however, say that this external *fabella* "indessen mehr dem Kneekehlmuskel gehört†." He also found this sesamoid very large in the Edentates, much larger than in other animals, a remark we can fully confirm.

Gruber‡ found in *Bradypus tridactylus* (the three-toed sloth) and in *Dasypus sexcinctus* (six-banded armadillo) no sesamoids in the heads of *gastrocnemius* or in the tendon of *popliteus*, but in *Myrmecophaga* there was in the tendon of *popliteus* a sesamoid articulating with the *capitulum fibulae*.

Gruber's result for *Bradypus* is somewhat surprising, for while there are no *fabellae* there exists, as Blainville first noted, a great *cyamella*; there is in addition a *lunula*. The point is well put by Mackintosh in his paper "On the Myology of the Genus *Bradypus*"§ that while there are no *fabellae* in *gastrocnemius*: "There is a large wedge-shaped *fabella* [i.e. *cyamella*] in the origin of this muscle [i.e. *popliteus*], the tibial surface of which is the larger and articulates both with the femur and the tibia."

We are, we think, reaching a stage of evolution in which lateral *fabella* and *cyamella* have coalesced and form a single large sesamoid. This large sesamoid may articulate with both femur and tibia as in *Bradypus*, or form a link between the femur and the *capitulum fibulae* as in the Anteater‖. Gruber, who never

* Compare what is stated later as to *Orycteropus*, p. 369.

† *Loc. cit.* p. 635. ‡ *Loc. cit.* p. 62.

§ *Proc. R. Irish Acad.* Vol. I, *Science*, 2nd Series, p. 528.

‖ This alternative, tibio-femoral or fibulo-femoral, link of the compound sesamoid might conceivably have bearing on the appearance of *fabellae* in *both* heads of *gastrocnemius*.

clearly distinguished between lateral *fabella* and *cyamella*, remarks* that in animals in which the *ossiculum* in *gastrocnemius externus* articulates on the head of the fibula, *popliteus* springs from the knee-capsule and from the *capitulum fibulae* not from the femur itself. We have indeed lost the distinction between the two sesamoids although we shall sometimes continue to speak of the *cyamella* as the single large sesamoid of the Edentates, because it is peculiarly associated with the *popliteus*†. The transition stage of the lemuroids in which *fabella* and *cyamella* were coming into close contact now provides much suggestiveness, which will be again emphasised when we come to the Marsupials.

In Plate XXIII, Fig. 55 *a* the reader will find a photograph of *Tamandua tetradactyla*‡ with the *cyamella* attached to the popliteal tendon originating in the popliteal groove on the lateral face of the external condyle. It will be easy to see how a not too careful articulator can mount this sesamoid on the surface of the external condyle. The *cyamella* itself articulates on the lateral side with external articular surface of the tibial head. This head projects over and downwards on the head of the fibula, which does not articulate with the femur. The large *cyamella* bears some trace of being a possible compound, i.e. *cyamella* and *fabella*. We have found this *cyamella*—without *fabella*—in:

Bradypus tridactylus (two mounted specimens), *Bradypus arctopithecius*, *Choloepus hoffmanni*, *Choloepus didactylus* (two specimens where the *cyamella* appears to be almost in the position of an anterior *lunula*), *Myrmecophaga tamandua*, *Manis dalmanni* and *Manis javanica*.

There is, we think, no doubt that in *Manis* we are dealing with a *cyamella* and not a lateral *fabella*. Owen, however, speaks of this sesamoid in *Manis* as an external *fabella* (*loc. cit.* p. 409). Windle and Parsons in their paper "Myology of the Edentates"§ remark under *gastrocnemius*: "It is interesting to notice that all observers have recorded the absence of *fabellae* except in the *Orycteropidae*" (p. 1004). This statement seems to overlook Owen and Gruber's statements as to *Manis*, and Macalister's with regard to *Tamandua* and *Myrmecophaga didactyla*.

In his paper‖ on the "Myology of *Bradypus tridactylus*" Macalister writes of the heads of *gastrocnemius*, and says that there are no sesamoids in the origins of the muscle as there are in *Tamandua* and *Myrmecophaga didactyla*. He then continues that Meckel refers to the Aï as possessing one of these in its origin and in the *Megatherium* one of these appears to have existed for the outer head of the *gastrocnemius*. Later referring to *popliteus* Macalister states correctly that the

* *Loc. cit.* p. 65, (15).

† It would be better to speak of it as the parafibula, except that in certain cases it has changed its position so as to articulate with the tibia.

‡ Macalister in his "Myology of *Bradypus tridactylus*" (*Annals of Natural History*, 1869, p. 15) speaks of the large sesamoid in the tendon of *popliteus* and states that it does not seem to occur in *Tamandua* (!).

§ *Proc. Zool. Soc. London*, 1899, p. 990.

‖ *Annals of Natural History*, 1869, p. 15.

sesamoid, a large one, of *Bradypus* lies in its tendon*, and then says that *popliteus* contains no sesamoid in *Tamandua*, *Orycteropus* and *Dasypus*. We feel sure he is wrong in the first case, the sesamoid of *Tamandua* is a *cyamella*. He is right about *Dasypus*. Owing to the kindness of Sir Arthur Keith it was possible for us to dissect a knee-joint of *Dasypus sexcinctus*; no *fabellae* were found, and only a cartilagenous thickening of the popliteal tendon, which it was not even possible to consider as a hemisesamoid†. There is no doubt, we think also, that the sesamoid in the fossil *Megatherium cuvieri* in the museum of the Royal College of Surgeons is a *cyamella* not a *fabella*. This fossil possesses also a very fine lateral anterior *lunula*: see Plate XXV, Fig. 66. Owen makes no reference to it, and we have not found any mention of it elsewhere. This sesamoid like the *cyamella* rests on the fibular articulating surface, but we consider that it would have been in the anterior border of the lateral semilunar. The so-called "fabella" is a parafibular sesamoid, it does not lie in the popliteal sulcus, but the tendon of popliteus would pass directly to it; it is not on the posterior surface but the lateral surface of the external condyle. It represents really something anterior to the differentiation of parafibula into *cyamella* and *fabella lateralis*.

Windle and Parsons' statements as to the sesamoids of the Edentates, namely that the

(*a*) *Bradypodidae* have no *fabellae* but a *cyamella* (p. 1010),

(*b*) *Myrmecophagidae* have no *fabellae* but a *cyamella* (p. 1011),

(*c*) *Manidae* have no *fabellae* but a *cyamella* (p. 1013),

(*d*) *Dasypodidae* have neither *fabellae* nor *cyamella* (p. 1012),

seem to us to represent the facts.

With regard to the *Orycteropidae* there appears to be some diversity of opinion. Humphry in his paper on the "Myology of *Orycteropus capensis*‡ (Aard Vark)" states that the three heads of *gastrocnemius* arise from the outer and inner condyles of the femur and from the head of the *fibula*, [*? soleus*], and that the two former have both sesamoids, i.e. *fabellae*. Galton in his "Myology of *Orycteropus capensis*"§ mentions only a sesamoid in the outer head of *gastrocnemius*, and cites Owen (*Anatomy of Vertebrates*, Vol. II. p. 409). He speaks of this *fabella* as "behind the outer condyle of the femur," and seems to think it corresponds exactly to what Meckel observed in *Choloepus didactylus* "though not mentioned by him in connection with this [*gastrocnemius*] or any other muscle." This is overlooking what Meckel himself wrote about the closer association of the sesamoid with the *popliteus* (see our p. 367)‖. And if *Orycteropus* really resembled *Choloepus*, then it would be a

* The drawing of the skeleton of *Bradypus tridactylus* in Parker and Haswell's *Text-book of Zoology*, p. 529, shows the *cyamella* and the *lunula*.

† In one mounted specimen of *Dasypus sexcinctus* facets on the condyles were observed which some might interpret as vestiges of *fabellae*.

‡ *Journal of Anatomy and Physiology*, Vol. II. p. 313.

§ *Trans. Linnean Soc.* Vol. XXVI. p. 594.

‖ Mackintosh ("Muscular Anatomy of *Choloepus didactylus*," *Proc. Royal Irish Academy*, Vol. II. 1875—77, p. 75) states definitely that *popliteus* has a large *fabella* [i.e. *cyamella*] in the tendon of origin.

cyamella and not a *fabella* with which we are concerned*. The maximum confusion seems to be introduced by Macalister's paper of 1875†. Writing of *popliteus* in *Chlamydophorus* he says: "I found no sesamoid bone in its tendon, as it lies under the external lateral ligament, nor does Hyrtl mention the existence of one. In the three-toed Sloth there is a large sesamoid bone. It is similar in *Choloepus*. There is no sesamoid bone in the *Orycteropus*, but there is in *Cyclothurus* and *Rholidotus*. There does not appear to be one in *Myrmecophaga* or in *Tamandua*" (p. 266). The latter statement is incorrect. The statements as to *Bradypus* and *Choloepus* are correct, but Macalister adds *Chlamydophorus* to *Orycteropus* as without *cyamella*.

Turning to *gastrocnemius*‡ he says that *Chlamydophorus*§ has the usual two heads neither of which possesses a sesamoid. There are no sesamoids in *Bradypus*, but Meckel found one in his specimen [we have already seen that Meckel found a popliteal not a *gastrocnemic* sesamoid]. No sesamoids exist in *Choloepus didactylus*. In *Orycteropus* there is a sesamoid bone in the outer head. [Humphry says he found a sesamoid in *both* heads!] A similar one exists in *Cyclothurus* and *Tamandua*. [This last statement is incorrect with regard to *Tamandua*, it may be correct but we have found only the *cyamella* in three limbs of *Cyclothurus*. If it were correct *Cyclothurus* would be the one Edentate that has both *cyamella* and lateral *fabella*.] As *Chlamydophorus* has neither *fabellae* nor *cyamella*, it must be classed with *Dasypus sexcinctus* rather than with *Orycteropus*.

Putting aside Humphry's statement with regard to *Orycteropus* which may have referred to an anomalous specimen, we have no case in which both *fabellae* occur in the Edentates. Even if Macalister's statement as to *Cyclothurus* be correct, we have only one case in which both *cyamella* and lateral *fabella* are present, and only two cases in which lateral *fabellae* have been recorded at all. There are two cases, *Dasypus* and *Chlamydophorus*, in which neither *fabellae* nor *cyamella* appear.

Unfortunately we have been able to examine relatively few moist knee-joints of the Edentates, and have had to trust largely to other investigators and mounted specimens. This is especially undesirable in the case of *lunulae*. We know, however, that they certainly exist in the Edentates, for example in *Megatherium cuvieri*, and in *Bradypus sexcinctus*, which have anterior *lunulae*, and in *Prodontes gigas* which has a mesial posterior *lunula*. We also think it safe to say that the specimen of *Orycteropus* examined by us (there were alas! no sesamoids *in situ*) had purely cartilaginous semi-lunars, i.e. no *lunulae*.

* The mounted specimen of *Orycteropus capensis* in the Museum of the Royal College of Surgeons has what seems to us a *cyamella* or parafibular sesamoid, but it is mounted on top of the external condyle! In the mounted specimen of *Manis aurita* the *cyamella* or parafibular sesamoid rests in what we think a much more natural position, i.e. between lateral condyle and head of fibula. It has been slightly separated from the latter for the sake of photography: see our Plate XXV, Fig. 67.

† "Anatomy of *Chlamydophorus truncatus* with Notes on the structure of other species of *Edentata*," *Trans. Royal Irish Academy*, Vol. xxv. p. 219.

‡ *Loc. cit.* p. 267.

§ We have examined two specimens of *Chlamydophorus* and find no *fabellae* or *cyamella*. The tibia and fibula are distally fused, and the latter is the more important member.

The typical Edentate may be said to be characterised by a large *cyamella* with probably one or more *lunulae*. The mesial *fabella* has practically no existence for this order, and in the few cases where the lateral *fabella* appears, it seems interchangeable with the *cyamella*. There is one group, however, *Dasypodidae* in which neither *cyamella* nor *fabellae* appear. The order of Edentates therefore *from our present standpoint* contains two fundamental groups one of which consists of species with non-sesamoidal knee-joint and the other of species with a sesamoidal knee-joint. The characteristic of the latter is the absence of the mesial *fabella* and the coalescence of lateral *fabella* and *cyamella* to form a large sesamoid which may act as an articulating link between the femur and one or other of the lower members of the limb.

(17) *Marsupials.*

It will be clear to the reader that the views expressed above on the affinities of genera, and orders based on the changes in the sesamoids of the knee-joint must receive strong confirmation, or failing it the *coup de grâce*, when we come to the Marsupials.

Historically we find the same vagueness as to the knee-joint sesamoids as in the Edentates. Meckel talks about the sesamoid of the kangaroo as if the kangaroo presented an "external fabella" only, and he makes a similar statement as to the opossum. Before Meckel Blumenbach had already noticed the external sesamoid in the *Didelphia* generally. Other fairly early notices of the external sesamoid are Pandir and d'Alton in *Didelphis cancrivora*, *Halmaturus elegans* and *H. giganteus* (Meckel also noticed it in *Halmaturus*), Cuvier and Laurillard in *Phalangista carifrons*, and Owen in *Myrmecobius* and some other marsupials. Owen describes the sesamoid as above and behind the external condyle of the femur*.

Gruber while still speaking of the lateral sesamoid of the Marsupials, as if it were directly comparable with the external *fabella* in other orders, gives useful information on its muscular relations. In *Didelphis philander* he tells us it is in the tendon of *gastrocnemius externus*, but articulates with the head of the fibula. The same statement applies to the sesamoid in *Dasyurus viverrinus*. Again in *Phalangista vulpina*, the external sesamoid articulates with the *capitulum fibulae* and is united to the fibula "durch eine schlaffe Gelenkkapsel und durch ein starkes Ligament." It is "fiddle-shaped" and with its thicker upper end the tendon of origin of *gastrocnemius externus* is united ("verwachsen"). In these three genera the external ossicle has no relation to the knee capsule and *popliteus* does not spring from the external condyle of the femur, but from the knee capsule and the *capitulum fibulae*†. None of the above has any internal *fabella*. These three species give very typical marsupial sesamoidal systems. They are cases really in which *fabella* and *cyamella* have coalesced and we do not believe it possible to speak of merely the "lateral" *fabella* existing. The remaining two cases, if Gruber has interpreted them rightly, seem to us more or less aberrant variants of the main marsupial type. They are:

* *Anatomy of Vertebrates*, Vol. II. p. 358. † *Loc. cit.* p. 59.

Hypsiprymnus, which has no mesial *fabella*, but (according to G.) a large lateral *fabella* articulating with external condyle*, and no sesamoid in the *popliteus*; and:

Halmaturus where (according to G.) *both fabellae* exist; the lateral articulates with external condyle and is united to tendon of *gastrocnemius*, but it had also a short ligament to the external semilunar. It was "hornformig."

We failed to find a mesial *fabella* in two specimens of *Halmaturus* (sp. ?). The species of *Halmaturus* is also not stated by Gruber.

According to Meckel *Halmaturus* (sp. ?) has only the lateral *fabella* and the same statement is made by Pandir and d'Alton for the species *H. elegans* and *H. giganteus*.

The only other record of a mesial *fabella* in the marsupials that we have come across is due to Windle and Parsons†, discussing the anatomy of *Macropus rufus*. They state that the inner head of *gastrocnemius* had also in the older specimen a *fabella* of much smaller size than that of the outer head and differently situated. "This *fabella* lies below the origin of the muscle from the back of the condyle of the femur and between the muscular fibres and that bone upon which therefore it plays."

We have examined a considerable number of mounted specimens of *Macropus*, the lateral sesamoid was almost invariably preserved but we have failed in any case to find a mesial orthosesamoid preserved‡. It is possible that the articulator in all these cases overlooked the smaller mesial *fabella*, but it is equally conceivable that the mesial *fabella* in Windle and Parsons' case was a hemisesamoid§.

Before discussing further other contributions to our knowledge of the knee-joint in the Marsupials, we wish to draw the attention of the reader to Plate XXIX, Fig. 78. The photograph on the right exhibits the lower limb bones of *Phascolomys*, the wombat. In this specimen there are no sesamoids at all, the large lateral sesamoid is seen as a fibular crest and is fused to the bone. The femur articulates with both tibia and fibula, and the head of the fibula comes in contact with the femur above the lateral condyle where the lateral *fabella* is often placed by the articulator. The limb to the left of the plate is that of *Sarcophilus*, the Tasmanian Devil. Here again the femur articulates with both tibia and fibula. The fibula has a parafibula precisely like that of *Phascolomys*. The summit of the fibula touches the femur above the lateral condyle, and there may even be a pit (as in the sea-otter; see our Plate XIX, Fig. 40) for its reception‖. The reader may say: "But this is the large external *fabella* of the marsupial myologists." It is and it is not; it really consists of two sesamoids, the lateral *fabella* and the *cyamella*; *they are in this case fused together and fused to the head of the fibula.*

* Confirmed by us on an articulated specimen of *Hypsiprymnus murinus*.

† *Journal of Anatomy and Physiology*, 1897, p. 128.

‡ *Macropus rufus*, *Macropus* (sp.), *Macropus agilis*, *Macropus major*, *Macropus bennettii*, etc.

§ At the same time we note that without finding a mesial *fabella*, we have found a cartilaginous thickening—hardly a hemisesamoid—in a specimen of *Macropus agilis* and in one of *Dendrolagus*, the tree kangaroo.

‖ As for example in the mounted specimen in the British Museum (Natural History).

Plate XXIX, Fig. 79, gives a second wombat; here the big sesamoid is not fused to the head of the fibula, but it nevertheless consists of the associated *lateral fabella* and the *cyamella* fused together, a true parafibula. The limb has had to be extended to separate the sesamoid from articulating on the femur. The reader should now examine our drawing after Pfitzner in Plate V, Fig. 13, of this memoir, where the muscular attachments have been entirely removed; he will (as a careful study will show him in the present plates) be able to detect the two portions of the associated bone, which become respectively the lateral *fabella* and *cyamella*. These two portions of the great marsupial sesamoid of the knee-joint are also quite distinguishable in *Phalangista*. See our Plate XXX, Fig. 80.

While the hypothesis that lateral *fabella* and *cyamella* are the product of the breakdown of the parafibula finds much support from the fact that specimens occur in which these sesamoids are actually fused together and to the fibula*, we have to consider how far this hypothesis fits the myological facts and tends to throw light on Gruber's description of the lateral *fabella* in man as a muscular "Knotenpunkt." We have above all to ask whether the Marsupials themselves provide evidence in favour of this break-up of a parafibula. One of the most noteworthy facts in this respect is the occurrence in certain marsupials of a hemisesamoidal *cyamella*, this articulates on the head of the fibula, and upon this *cyamella* articulates the lateral *fabella*, which in its turn is in contact with the upper part of the condyle of the femur.

A specimen of *Macropus* (sp. ?) sent to us from Australia shows a lateral *fabella* —an orthosesamoid—and a hemisesamoid *cyamella*. The specimen consisted only of the lateral branch of the *gastrocnemius* with its orthosesamoid, the hemisesamoidal *cyamella* and the external semilunar cartilage all attached, these for the left hind limb. It was thus not possible to draw the knee-joint showing these in their natural position. It is not, however, difficult to understand their relations to other structures in the knee-joint, if a comparison be made between these drawings and those of other members of the *Marsupialia*. See Plate XXXIII, Figs. 83 *a* and 83 *b*.

In a museum specimen of *Macropus agilis* we found also this hemisesamoidal *cyamella*, while the lateral *fabella* was a large orthosesamoid. In the mesial head of *gastrocnemius* there was a cartilaginous nodule, possibly a hemisesamoid.

In another mounted specimen of *Macropus rufus* the large lateral sesamoid was simply wired on to the femur, a process which completely fails to indicate the sesamoid system as a link between femur and *capitulum fibulae*. Our Plate XXXIV, Figs. 87 and 88, shows museum specimens of *Macropus bennettii* and *Perameles lagotis* in which this sesamoid is simply pinned on to the femur†. Our Plate XXXV, Fig. 90, gives what appears to us to be the position of this sesamoid in Bennett's Wallaby; it has very clearly no relation to the position as indicated in Figs. 87 and 88.

* We have never in examining several hundred knee-joints found a case in which the sesamoids were fused to the femur, and the only recorded case of ankylosis that we have met with is that reported by Gruber in a specimen of *Myogale*.

† We have also found the parafibular sesamoid treated in this manner in the case of *Dendrolagus ursinus*, where the sesamoid is generally similar to that of *Macropus bennettii*, and again in *Macropus major*.

But a preparation in the Museum of the Royal College of Surgeons* shows far more clearly the whole arrangement of a knee-joint of this character. We owe photographs of it in anterior and lateral aspects to Mr S. Steward (see our Plate XXXIV, Figs. 89 *a* and 89 *b*). They show the sesamoid mounted vertically on the top of the fibula and connected proximally by a tendon to the femur above the lateral condyle. The sesamoid itself is a true parafibula and does not articulate with the femur at all, as is well indicated by the anterior aspect†. In the cases of *Bettongia gaimardi*, *Hypsiprymnus murinus*, and *Myrmecobius fasciatus*, we think there is little doubt that we are dealing with a parafibular sesamoid and not with a simple lateral *fabella*.

Windle and Parsons‡, who have treated at length of the anatomy of *Macropus rufus*, state that

> the tendon of the *popliteus* contains a very strong fibro-cartilaginous *fabella*, which is placed above the head of the fibula to which it is attached, and on which it moves freely. It is with this *fabella* that the large gastrocnemic *fabella*, to be more fully described in another section, articulates.

And again:

> The outer head of the *gastrocnemius* possesses a remarkably large *fabella* already mentioned as articulating with that of the *popliteus*. This was crescentic in shape [see our Plate XXXV, Fig. 90] and in the older specimen measured 2·5 cms. from one corner to the other.

To the muscular attachments as noted by our authors we shall return later. We are here only concerned with making the point that the *cyamella* portion of the associated femoro-fibula sesamoidal link of the Marsupials may in certain species have degenerated into a cartilaginous§ mass—a hemisesamoid. In cases where this hemisesamoidal *cyamella* disappears entirely the whole purpose of the *fabella-cyamella* system is lost to sight, and we are provided with a detached sesamoid on the lateral condyle associated with a system of muscular attachments which formerly were associated with the *capitulum fibulae*.

It seems to us that in the *Marsupialia* there has been written the complete history of the lateral sesamoids of the knee-joint. We have:

(i) No sesamoids at all but a fibular crest, a continuous part of the fibula (some specimens of Wallaby).

(ii) The parafibula is still fused with the fibula but shows signs of splitting into two parts, *fabella* and *cyamella* (certain specimens of Tasmanian Devil).

(iii) The parafibula separates off as a large lateral sesamoid being an articulating link between femur and fibula. The two parts *fabella* and *cyamella* are still united (this is the general rule in the wombat).

(iv) The large lateral sesamoid breaks up into two components, the *fabella* and the *cyamella* (this is the case in *Didelphys*), or:

(v) The *cyamella* becomes a mere hemisesamoid (*Macropus rufus*), or:

* See note †, p. 373.

† This parafibular sesamoid is not shown in the figured skeleton of the wallaby on p. 130 of Vol. x. of the *Cambridge Natural History*, nor is it referred to in the text; even with Galen of such bones "non est necessarium hoc opere pertractare"!

‡ "On the anatomy of *Macropus rufus*," *Journal of Anatomy and Physiology*, 1897, p. 123 and p. 128.

§ We much doubt its being "fibro-cartilaginous" as Windle and Parsons state. A section of the *cyamella* in *Macropus* should be examined as soon as possible microscopically.

(vi) The *cyamella* disappears entirely or almost entirely, being merely represented by a thickening of the popliteal tendon at the position where definite ossification occurs; noted in some cases of *Didelphys*.

We have already illustrated (i) and (ii) from individual specimens of the wombat and the Tasmanian Devil. (iii) is the more general case of both Tasmanian Devil and wombat. It is also the case of *Dasyurus macrourus*, *Dasyurus ursinus*, *Perameles nasuta* and in some instances of *Phalangista vulpina**.

Didelphys virginiana is typical for (iv). Our sketch, Plate XXXIII, Fig. 84, shows the relative position of *fabella* and *cyamella*—they form together when the cut ends of *gastrocnemius* are reunited an articulating link between the head of the fibula and the external condyle. The reader must remember that in the natural position the limb is much more flexed than in the sketch.

(v) Our drawing, Plate XXXIII, Figs. 83 *a* and 83 *b*, indicates in *Macropus* the *fabella* with hemisesamoidal *cyamella*.

Our drawings of *Phalangista vulpina* and *Phascolarctus* indicate the nature of the lateral side of the knee-joint when there is a single parafibular sesamoid which has often been described as a *fabella lateralis* (see Plate XXXIII, Figs. 85 and 86).

In mounted specimens of *Bettongia gaimardi*, *Hypsiprymnus murinus*, *Peragalea lagotis*†, *Myrmecobius fasciatus*, we found only a sesamoid attached to the lateral condyle.

Fossil Marsupials showing Parafibulae.

Prothylacinus patagonicus *Cladosictis lustratus* *Diprotodon australis*
From the Santa Cruz Beds, after Sinclair after Shaw and Parker

The fossil *Diprotodon australis*‡ belongs apparently to class (iii), and shows according to Professor Stirling's restoration a large parafibula which would only

* Our photograph, Plate XXXIII, Fig. 85, shows the combined sesamoid in *Phalangista*.

† In specimens of *Peragalea lagotis* and *P. leucura* the fibula did not articulate with femur, there was no ossification in the popliteal tendon, but the lateral *fabella* was present.

‡ See Parker and Haswell, Vol. II. p. 602, Fig. 1229. The fossil Patagonian Marsupials reconstructed by Sinclair show fibular crests (or parafibulae?). See his plates of *Cladosictis lustratus* and *Prothylacinus patagonicus*. He gives no details, however, on this point. He does say that the patella is ossified in the latter. See *Proc. American Phil. Soc.* Vol. XLIV. pp. 73—81, Philadelphia, 1905.

come in contact with the femur on extreme flexure. This figure throws much light on the position of the large lateral sesamoid in the Tasmanian Devil (see our Plate XXXIV, Figs. 89*a* and 89*b*) and in *Halmaturus* (see our Plate XXXV, Fig. 90).

In *Pseudochirus peregrinus*, one of the phalangers, the lateral condyle of the femur articulates with the fibula and both lateral *fabella* and *cyamella* are present. The fibula was prolonged upwards and posteriorly into a wing on the top of which the lateral *fabella* was found. The *cyamella* is a small nodule or inner notch of fibula. The arrangement is somewhat similar to that of *Didelphys*.

If the hypothesis developed in this paper be correct, namely that the lateral *fabella* and *cyamella* are vestiges of the break-up for the parafibula of an early mammalian evolutionary stage, we ought to find the lateral head of *gastrocnemius*, in whole or part, and the tendon of *popliteus* transferred to the *capitulum fibulae*. We shall turn shortly to the myology of the Marsupials and inquire whether there is any support for such a view.

One of the noteworthy points of the Marsupials and Monotremes is the apparent absence of *lunulae* which are so definitely present in many reptilian forms. We have failed to find them in *Ornithorhynchus*, or in the koala, wombat and a variety of kangaroos. The alternatives before us are: to suppose that both Monotremes and Marsupials have lost original *lunulae*, although they occur regularly or at least as anomalies in many placentals, or to suppose both to have originated in a reptilian form possessing a fibular crest or a parafibula, but no *lunulae*. A similar problem arises in the case of the Ungulates; we must suppose them to have completely lost all sesamoids of the knee-joint, and so thoroughly that they do not even appear as in man as anomalies, or we must conceive them descended from an ultimate form which has never possessed *parafibula* or *lunulae*.

(18) *On changes in the muscular attachments of the knee-joint.*

If we look upon the lateral *fabella* and the *cyamella* in Man as vestiges of an earlier structure which has now disappeared, it becomes of importance to consider the evolutionary history of the muscles attached to these sesamoids. Does it indicate a transition of their attachments from one bone to a second which is compatible with a transfer of the sesamoids from one bone of the lower limb to a second? We venture to think it does, and shall endeavour to place before the reader in this section our reasons for believing that the transfer of muscular attachments from the head of the fibula to the distal epiphysis of the femur has been accompanied by a transfer of the remnants of the parafibula to the lateral condyle of the femur.

The muscles with which we are principally concerned are *M. gastrocnemius* (in particular the lateral branch), *M. popliteus* and *M. plantaris*. Gruber has gone so far as to assert that the lateral *fabella* serves a useful purpose as a "Knotenpunkt" directing on the posterior surface of the lateral condyle the movements of *gastrocnemius externus, plantaris, semimembranosus* and occasionally *M. popliteus biceps*, when this anomalous supernumerary head of *popliteus* is present. Without accepting this interpretation of "usefulness" in a vestigial structure, we would

suggest that the "Knotenpunkt" nature of the lateral *fabella* is a phenomenon to be accounted for. We must seek for an important structure to which these muscles could have been originally attached, and which in a pre-vestigial stage had a use value*. The supernumerary head of *popliteus* arising in the *fabella*, connects the lateral *fabella* at once with the *cyamella* in the principal head of *popliteus* as possible parts of the same original structure. The next point is to consider what changes have taken place in the attachment of *popliteus*. There is no doubt that in the course of evolution *popliteus* has migrated from the head of the fibula to the lateral face of the external condyle. Take the *cyamella* with the popliteal tendon attached still both to the condyle and to this sesamoid and place it at the head of the fibula; place the lateral *fabella* on top of the *cyamella*; fuse them together and we have by no means a bad representation of the muscular attachments of the highly flexed marsupial knee-joint and their relations to the parafibular sesamoid. Fuse this parafibula to the head of the fibula and we have a rough representation of the muscles of the knee-joint of the Monotremes.

Let us examine first the Monotremes. *Popliteus* in *Echidna* arises from the extremity of the process at the top of fibula instead of from the external condyle. *Gastrocnemius* is a slender and delicate muscle arising from the internal condyle; this is really the internal head, and the external head appears to be replaced by *Soleus* which "arises by tendinous fibres from the posterior and outer margin of the singular process at the summit of the fibula." This is Mivart's account† of these muscles; he does not refer to *plantaris*.

Manners-Smith has dealt with *Ornithorhynchus*‡, and describes the knee-joint musculature as follows:

He tells us that *gastrocnemius* is a large muscle arising by two distinct heads, a larger from the fibular crest and a smaller from "the lower end of the femur." It does not fuse with the *soleus*. The femoral head, he states, is supplied by the external popliteal. *Soleus* arises from the fibular crest and from the shaft of the fibula. A *few* fibres only arise from the tibia. Manners-Smith contradicts Owen's assertion that a large proportion of the fibres arise from the tibia and asserts that it is certainly a mistake. He gives no information about *popliteus* or *plantaris*. Our drawing (Plate XX, Fig. 42) shows the external head of *gastrocnemius* having no attachment to the external condyle and the internal head attached to the internal condyle.

In Reptiles *popliteus* arises from the fibula§, probably also from a groove on the external condyle and when it exists from the parafibular sesamoid‖. According to

* As we have previously remarked we cannot consider a structure which has only an occasional and anomalous occurrence to have real "use" value; it can only be degenerating, i.e. vestigial.

† "Anatomy of *Echidna hystrix*," *Trans. Linnean Society, London*, 1866, p. 394.

‡ "Anatomy of *Ornithorhynchus*," *Proc. Zoological Society, London*, 1894, p. 707.

§ Mivart, "Myology of *Iguana tuberculata*," *Proc. Zoological Society, London*, 1867, p. 796. *Popliteus* arises from tibial aspect of head of fibula.

‖ See our p. 389, and the drawings of *Varanus*, Plate XXIV, Figs. 64 *a* and 64 *b*.

Macalister* *popliteus* in birds may be femoro-tibial or fibulo-tibial in origin, and the origin of *soleus* is also variable. There is of course the possibility that *popliteus* was not in the earlier stage largely fibular in its attachment, but acquired an attachment to the fibula in the marsupials. This appears to be the view of Parsons who writes†:

> In the koalas, the phalanger, the opossum and the Tasmanian devil, the great mobility of the fibula has produced the same effect on the *popliteus* that we noticed in the lemur, the fleshy part acquires an attachment to the fibula, while the tendon is converted into an extra external lateral ligament.

The question arises, however, as to whether the popliteal attachment to the fibula is not the earlier stage and thus the arrangement we find in the lemur would be vestigial. When the parafibula broke up, it is conceivable that *popliteus* carried off with the *cyamella* the whole or a portion of a fibulo-femoral ligament, i.e. it is not the tendon of *popliteus* converted into a ligament, but a ligament which has become the tendon of *popliteus*.

The point has been studied also by Gordon Taylor and W. V. Bonney‡. They take the view that the ultimate origin of the *popliteus* was fibular; that as movement between the tibia and fibula ceased to exist the upper origin of *popliteus* ascended to the external femoral condyles, but that in the lemur where movement between tibia and fibula has reappeared the *popliteus* appears to be descending to its primitive fibular attachment. The lemur problem is a question either of reversion or of vestige. The question of whether the fibula attachment of *popliteus* is the original attachment or an acquired one seems worthy of much more study especially in relation to the origin of the lateral knee-joint sesamoids, in particular to the muscular attachments of the parafibula in Reptiles.

Passing from the Monotremes to the Marsupials it becomes difficult to appreciate the contributions of various writers to the subject, for they invariably speak of the parafibular sesamoid as if it were a large *fabella* in the lateral head of *gastrocnemius*. Thus we find it hard to interpret the following statement of Macalister with regard to the myology of the Marsupials§:

> The *gastrocnemius externus* in the Wombat, Giant Kangaroo, Sarcophilus and Bennett's Kangaroo arises from the sesamoid bone at the back of the external condyle of the femur and is inserted into the tendo Achilles, etc.

It seems to us extremely difficult to speak of the parafibular sesamoid, which may even be fused to the fibular head, in the Wombat and Tasmanian Devil (see our Plate XXIX, Fig. 78) as a sesamoid at the back of the external condyle! The same difficulty occurs in the Wallaby (Bennett's kangaroo, or Halmaturus): see

* "On some Points in Bird Myology," *Proc. R. Irish Academy*, Vol. II. 1875—77, p. 57.

† "Joints of Mammals compared with those of Man," *Journal of Anatomy and Physiology*, Vol. XXXIV. p. 313, 1900.

‡ "Homology and Morphology of Popliteus," *Journal of Anatomy and Physiology*, Vol. XXXIX. p. 35.

§ "Myology of the Wombat (*Phascolomys wombata*) and the Tasmanian Devil (*Sarcophilus ursinus*)," *Annals of Natural History*, 1870, p. 19 of our offprint.

our Plate XXXV, Fig. 90. We are dealing in all these cases with something much more closely allied to fibula than to femur.

Again:

The *popliteus* in the Wombat and Sarcophilus is very large but thin, arising from the upper third of the back of the fibula, and inserted into the lower two-thirds of the back of the tibia. A few fibres of this muscle in Sarcophilus are attached to the sesamoid bone in the outer head of *gastrocnemius*. This muscle is smaller in the Giant Kangaroo, but in this and *Macropus bennettii* its origin is purely sesamoid*

These two passages taken together suggest that both in the Great Kangaroo and the Wallaby *gastrocnemius externus* and *popliteus* arise from a *fabella* or sesamoid at the back of the external condyle. They do not suggest that they arise from a parafibular structure, which may be opposed rather laterally than posteriorly to the femur (see our Plate XXXIV, Figs. 89 *a* and 89 *b*), and which is itself united by a stout band to the femur above the external condyle, this band corresponding in the placentals to the tendon of *gastrocnemius externus*.

Macalister's statements about other muscles are clear. He tells us that *gastrocnemius internus* arises from the internal condyle and joins the tendo Achilles... it is larger than the *externus* in the Wombat, but smaller in the Tasmanian Devil and in all species examined was without sesamoid.

Soleus has a fibular small origin in the Wombat, but as usual has no tibial heads. The *soleus* is fibular also in Tasmanian Devil, Opossum, Phalanger and Giant Kangaroo.

The *plantaris* in the Opossum and Perameles is small but separate from the outer condyle to the outer side of the head, but no *plantaris* exists in the Wombat. A distinct small muscle in the Tasmanian Devil passes from the back of the external lateral ligament and from the head of the fibula down the internal side of the calcaneum and into the plantar fascia†.

In a *Macropus sp.* examined by one of the present writers the parafibular structure consisted proximally of an orthosesamoid from which *gastrocnemius externus* arose, this sesamoid articulated on a hemisesamoid which was linked by a tendon to the lateral face of the external condyle and from which *popliteus* arose; this hemisesamoid rested on the *capitulum fibulae*. *Plantaris* was not identifiable in our specimen. The orthosesamoidal part of this parafibular structure is what is termed by most writers the *gastrocnemial* lateral *fabella*, and the hemisesamoid is what we term the *cyamella* portion of the parafibular structure. *Popliteus* is not attached to the orthosesamoidal part of the structure at all. Interpreted in our own terms we see that this practically agrees, except with regard to *plantaris*, with what Windle and Parsons report of *Macropus rufus*‡:

* *Loc. cit.* p. 17 of our offprint.

† Macalister, *loc. cit.* p. 19 of our offprint.

‡ *Journal of Anatomy and Physiology*, Vol. XXXII. p. 123, 1897. They say of *plantaris*, that the upper cornu of gastrocnemial *fabella* as well as external condyle give rise to *plantaris* which is thus a double-headed muscle.

The tendon of the *popliteus* contains a very strong fibro-cartilaginous fabella, which is placed above the head of the fibula to which it is attached, and on which it moves freely. It is with this fabella that the large gastrocnemial fabella articulates.

In another paper* Parsons notes that in the koala, the phalanger, the opossum and the Tasmanian Devil the fleshy part of *popliteus* has acquired an attachment to the fibula while the tendon is converted into an extra external lateral ligament.

According to the views expressed in this paper it may be anticipated that the primitive marsupial would have a fused parafibula, and accordingly the later tendon of *popliteus* would be a femoro-fibular ligament and *popliteus* would arise from the fibula. From this standpoint it is not surprising that on the break-up of the fibular crest certain portions of *popliteus* should still be attached to the fibula rather vestigially than as an acquirement.

We dissected a specimen of Koala (*Phascolarctus cinereus*) and found the parafibular sesamoid articulating with the fibula: see our Plate XXXII, Figs. 82 *a* and 82 *b*. The fleshy portion of *popliteus* was attached to the fibula and the popliteal tendon ran from the antero-proximal corner of fibula to popliteal groove of external condyle. Also tendinous fibres from the fleshy portion on fibula cross the posterior surface of the fibula and mingle with the tendinous fibres of *gastrocnemius*, which run from the sesamoid to the external condyle.

No *plantaris* was discoverable. Young in his paper on the "Muscular Anatomy of the Koala†" does not refer to *popliteus* or *plantaris*, but states that *gastrocnemius* is well developed and arises by two heads in the external of which a *fabella* exists. We again urge that this parafibular sesamoid should not be described as a *fabella*. We think that for the whole genus *Macropus* we ought to consider that we are dealing with a parafibular structure and not a lateral gastrocnemial *fabella*. We have not been able to examine a case of *Petrogale xanthopus*, but Parsons' description seems in accordance with the above interpretation, if we replace his term *fabella* by parafibular sesamoid. Thus he says‡:

The knee-joint like the hip is never fully extended, the capsule is strengthened by strong lateral ligaments; the external lateral runs downwards and backward from the external condyle to the front of the head of the fibula; its anterior fibres are continuous with the origin *extensor longus digitorum*. The *popliteus* also performs the function of an external lateral ligament, since it is attached to the femur and fabella, and below to the external semilunar cartilage and the head of the fibula. During extreme flexion which is the usual position of the joint, the *fabella* articulates with the head of the fibula.

Judging from the knee-joints of other specimens of *Macropus* with which we are more familiar, we should believe that the parafibular sesamoid was always in contact with the fibula and that in cases of extreme flexure its head may come in contact with the postero-lateral border of the femur. We do not think that like an

* "Joints of Mammals compared with those of Man," *Journal of Anatomy and Physiology*, Vol. XXXIV. p. 313, 1900.

† *Journal of Anatomy and Physiology*, 1882, p. 237.

‡ "On the Anatomy of *Petrogale xanthopus* compared with that of other Kangaroos," *Proceedings Zoological Society, London*, 1899, p. 690.

ordinary *fabella*, it will be found to articulate with the articular surface of the lateral condyle. Turning to the muscular attachments Parsons states:

The *gastrocnemius* has the usual two heads, the outer of which has three origins: (1) a small head from the outer side of the patella, (2) larger from the same place and separated from the last by the external popliteal nerve, (3) from the large *fabella* over the external condyle and from the external semilunar cartilage. The inner head is normal in origin and has no *fabella* developed in it (p. 207).

The *soleus* is absent unless the origin of the *gastrocnemius* from the semi-lunar cartilage represents it. Meckel also describes it as wanting.

Plantaris comes from the external *fabella**.

Popliteus arises from the external condyle, partly from in front of the groove, and partly from the groove itself, which is deep and narrow and not adapted for the tendon to lie in†. The muscle also has an origin from the external semi-lunar cartilage and from the head of the fibula.

We see in this typical description of a marsupial knee-joint, *gastrocnemius*, *popliteus* and *plantaris* all associated with the parafibular sesamoid. We see the muscles which largely arise from the fibular crest in the Monotremes now attached to the parafibular sesamoid, and the fragments of the parafibula will be carried with them as *fabella* and *cyamella* to their ultimate tendons of insertion on the lateral femoral condyle. We begin to see now why the early human anatomists associated the lateral *fabella* with numerous muscles, why *popliteus* (with its anomalous head) passes to *fabella* as well as to *cyamella*; how Gruber could confuse these two sesamoids, and why he emphasised so strongly the "Knotenpunkt" character of the lateral *fabella*.

While *Macropus* seems to us to represent a later evolutionary stage of the parafibular sesamoid than *Phascolomys*, for in the Kangaroos we have various stages of disruption of the parafibula, while in the Wombat it may still be fused, the Phalangers form a more or less intermediate stage. They have a parafibular sesamoid of much the same form as the wombat's—quite different from that of the wallaby—and its form is seen to be clearly that of a fibular crest: see our Plate XXXIII, Fig. 85.

As Cunningham has shewn, *gastrocnemius* in *Phalangista maculata* springs from the posterior aspect of the fibula in its upper two-thirds, and *plantaris* from the parafibular sesamoid. Cunningham suggests that *gastrocnemius* is a compound muscle containing *soleus*‡. The musculature is much the same as in the Koala (*Phascolarctus cinereus*), except that in the latter we did not identify *plantaris*§, which is also said not to exist in the Wombat.

In *Perameles nasuta* we have the parafibular sesamoid, but the *Peramelidae* are transitional to the *Didelphyidae*. Parsons has pointed‖ out that "the popliteus

* That is the parafibular sesamoid.

† Any marked popliteal sulcus has disappeared in *Macropus*, see our drawing of the Wallaby, Plate XXXV, Fig. 90.

‡ *Challenger Report*, Vol. v. p. 40.

§ In the Koala there is a collatero-fibular ligament—the ultimate popliteal tendon connecting the parafibular sesamoid with the lateral condyle of the femur.

‖ *Journal of Anatomy and Physiology*, Vol. xxxiv. p. 313, 1900.

tendon instead of being converted into an extra lateral ligament is continuous with its fleshy belly as usual and patella is ossified*." According to Macalister† the *plantaris* in *Perameles* as in *Didelphys* is small and arises from the outer condyle.

Of the *Dasyuridae* Cunningham has dealt with *Thylacinus cynocephalus*: he states that *gastrocnemius* arises by two very large and powerful heads: (1) The outer head which contains the *soleus* is much the larger of the two and it arises by two parts, (*a*) by a fleshy process fixed to a large sesamoid bone placed upon the posterior aspect of the head of the fibula—clearly our parafibular sesamoid, and from a powerful ligament which binds the upper part of this bone to the back of the external condyle of the femur, (*b*) by a tendinous slip attached to the outer aspect of the external condyle of the femur. (2) The inner head springs not only from the back of the inner condyle but also from the whole breadth of the popliteal surface at the same level. *Plantaris* in *Thylacinus* arises in common with the outer head of *gastrocnemius* and remains fused with its under surface for a considerable distance.

We have already noted that in *Dasyurus macrourus* and *D. ursinus* the sesamoid is parafibular (see our p. 375). MacCormick‡ has studied the myology of the limbs of *D. viverrinus*. He states that the outer head of *gastrocnemius* "arises directly from the sesamoid bone over the upper end of the fibula, and indirectly by two strong ligaments which bind this bone to the femur, one being attached to the outer border of the popliteal space, and the other passing over the femur to be attached to the upper border of the patella§ and to the outer part of the *vastus externus* tendon." He does not refer to *popliteus* and *plantaris*.

Didelphys is the last marsupial to which we shall refer. It represents the later stage of the break-up of the parafibula. *Gastrocnemius externus* arises from external condyle and contains a sesamoid—the lateral *fabella*. *Popliteus* follows the course of this muscle in higher mammals, arising from a groove in the lateral face of the external condyle, and contains a *cyamella*. We found no association between *popliteus* and the gastrocnemial *fabella*. But the *cyamella* of the Opossum is wedged between the under surface of the lateral condyle and the head of the fibula (see our Plate XXXIII, Fig. 85), and when the *gastrocnemius* is replaced the *fabella* lies on the external condyle immediately above the *cyamella*. According to Parsons‖, however, the fleshy part of *popliteus* in the Opossum acquires an attachment to the fibula, while the tendon is converted into an extra lateral ligament. This did not seem to be the case in our specimen, the tendon of *popliteus* passed to the *cyamella*, which indeed might be looked upon as a parafibula, but unless that sesamoid was fused the tendon could hardly be treated

* It is not ossified in Kangaroos and Wallabies and Opossums, although a small bony element has been reported in certain cases.

† *Annals of Natural History*, 1870, p. 17.

‡ *Journal of Anatomy and Physiology*, Vol. XXI. p. 213, 1887.

§ The reader must remember that the patella is cartilaginous in all Marsupials but *Perameles* although the existence in certain cases of a small ossified kernel has been asserted. The attachment the fibular crest to the patella in the case of Platypus may also be borne in mind here.

‖ *Journal of Anatomy and Physiology*, Vol. XXXIV. p. 313, 1900.

as an extra lateral ligament in our case. We have already seen that the fusing* of the parafibular sesamoids is a possibility in the Marsupials, rare as it appears to be in the higher Mammals†. It is possible that in certain individuals, or even in one or other species of opossum the *cyamella* may as a vestige be fused to the fibula, and so the popliteal tendon become an extra lateral ligament.

Plantaris arises in *Didelphys* from the lateral femoral condyle‡.

It will be seen from the above that *Didelphys*, as far as the sesamoids of the knee-joint are concerned, is among the most advanced of the Marsupials. It stands at the opposite extreme to the Tasmanian Devil with its fused parafibula forming a fibular crest. The range marks the transfer of the *gastrocnemius*, *popliteus* and *plantaris* from fibula to femur and with them pass both lateral *fabella* and *cyamella* to the lateral femoral condyle.

(19) *Monotremes.*

We now come to the Monotremes and have to bear in mind the fundamental differences we have noted between the *Ornithodelphia* and the *Monodelphia*, in particular the existence of the parafibula, and the rearrangement of the muscular insertions of the posterior aspect of the knee-joint. We turn first to *Ornithorhynchus* (Platypus). Plate XX, Figs. 43 *a* and 43 *b* provide drawings of the femur and distal portions of tibia and fibula. The extraordinary development of the process at the distal end of the fibula will be at once noticed. From this fibular crest (see our Plate XX, Fig. 42) arises the lateral head of *gastrocnemius*, *plantaris* and *popliteus*. The patella is also attached to this expanded head of the fibula. This is well shown in the prepared specimen in the Museum of the Royal College of Surgeons. The internal head of *gastrocnemius* is attached above the internal condyle of the femur. There are no *fabellae*§, no *cyamella*, and what is more remarkable no *lunulae* (i.e. because the latter are very conspicuous in the Reptiles). Thus platypus verifies what we have observed in the Marsupials: namely that, with the appearance of the fibular crest, there is associated a disappearance of lateral *fabella* and *cyamella*. Accordingly we see in the Monotremes the three muscular attachments which are associated with the lateral *fabella* in Man (see our Part I, p. 161); those of the external head of *gastrocnemius*, of *popliteus* and of *plantaris*, are now associated with the fibular crest. Manners-Smith tells us that‖ *soleus* also arises from the expanded head of the fibula, but partly too from the shaft of the same bone.

By the kindness of Sir Arthur Keith one of us was able to dissect the knee-joint of *Echidna*. There were no signs of any *fabellae*. The popliteal muscle followed

* We must not be understood as suggesting that this "fusing" is something which may occur anomalously as in an adult individual; it is specific in the case of the two parts of the parafibula, and vestigial when the parafibula is ankylosed with the fibula.

† See our p. 373, first footnote.

‡ This was so in our specimen, and Macalister says so also: see *Annals of Natural History*, 1870, p. 17.

§ Meckel, *loc. cit.* pp. 634—635, had already noticed the absence of the *fabellae* in Platypus ("Schnabelthier"). Gruber has not investigated the Monotremes.

‖ "Anatomy of Ornithorhynchus," *Proc. Zool. Soc. London*, 1894, p. 707. He also notes the fibular origin of *gastrocnemius externus*, but says nothing of *popliteus* or *plantaris*.

much the same course as that shown in the drawing of *Didelphys*, but it lacked any *cyamella* where it passed over the head of the fibula; it arises as in *Ornithorhynchus* from the fibular crest and not from the external condyle of the femur. The expansion of the fibula is not quite so marked as in *Ornithorhynchus*, but from it arises not only *popliteus*, but the larger or external head of *gastrocnemius*. The internal head is much smaller and arises from the internal condyle*.

(20) *Centre of Ossification of Fibular Crest in Monotremes.*

In order to grasp how the fibular crest can be separated from the fibula and ultimately be found disintegrated as vestigial sesamoids attached to *gastrocnemius*, *popliteus* and *plantaris*, it is of some importance to determine whether this process at the head of the fibula has a separate centre of ossification.

Parker and Haswell† appear to have no doubt on the matter; writing of the Monotremes they say:

> The fibula has at its proximal end a remarkable compressed process which ossifies from a separate centre and greatly resembles the olecranon of the ulna.

Through the great kindness of Dr Wm. Colin Mackenzie of the Australian Institute of Anatomical Research, Melbourne, we were able to obtain sections of the fibula both of an embryonic and a young platypus. He also sent us the section of the parafibular sesamoid of a young wombat which had just left the pouch. Fig. A, on p. 385, indicates the bone of the young platypus set against that of an adult platypus to give the relative sizes. The length of the former animal from tip of bill to cloacal sphincter was 31 cms. as against the similar length of a fully developed adult of 44 cms. The length of the embryonic platypus was 13 cms. Fig. B shows the position of the sections Dr Mackenzie made of the embryonal and young platypus.

Taking the section A of the embryo (see our Plate XXXVII, Fig. 96), we have periosteum at the proximal end of the fibular process (top of plate), then follows a small area staining yellow and suggesting ossification, then an area of cartilage with every appearance of ossification starting, and then cartilage again. This certainly seems to accord with a centre of ossification in this fibular process or crest. In the young platypus in the corresponding section A we found no cartilage at all but bone covering the areas where in the embryo we had found cartilage and ossification taking place. It was therefore probably not young enough to show an intermediate partly ossified stage. On the other hand the section towards the top of the crest (Plate XXXVII, Fig. 95) shows fairly numerous Haversian canals which become less frequent or cease altogether as we pass down the section, to become frequent

* Mivart ("Anatomy of *Echidna hystrix*," *Trans. Linnean Soc.* 1866, p. 394) mentions only what appears to be the inner head of *gastrocnemius*, and probably our outer head is his *soleus*, which like his *popliteus* arises from the process at the top of the fibula.

† *A Text-book of Zoology*, Vol. II. p. 489, London, 1897. The comparison of the olecranon with the tibial crest seems more reasonable, the fibular crest should be homologous with a radial rather than an ulnar process. The fibular crest of the Monotremes or that of certain Marsupials has little general resemblance to the olecranon, while the tibial crest in birds (*Colymbus*) or the tibia plus patella (as in *Hesperornis*) is very similar to the olecranon in appearance.

again and in a different direction as we reach the shaft. Their arrangement seems to us not inconsistent with a separate centre of ossification for the fibular crest.

Our view differs from that of Dr Mackenzie and Mr W. Owen, histologist to the Australian Institute. They speak of section A of the embryo as if it were wholly cartilaginous. It seems to us as if a portion of this superstructure of the fibula does show ossification starting (Plate XXXVII, Fig. 96). The articular process of fibula, section B of Fig. B below, has a distinct centre of ossification in the embryo. Further the section at C also shows ossification and Dr Mackenzie tells us that

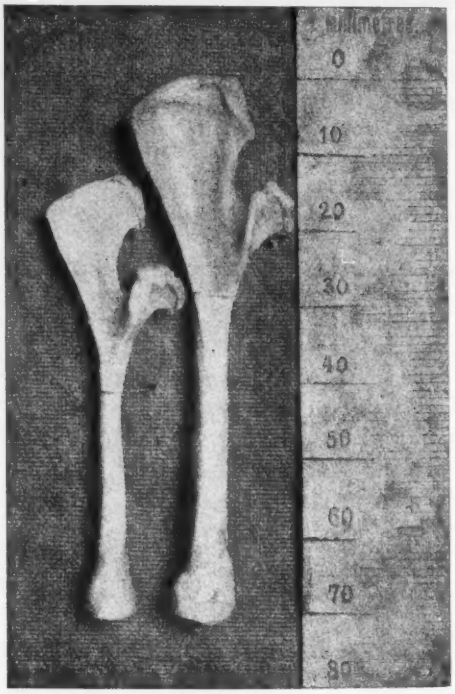


Fig. A.

Fig. B.

there is one at the distal end (section not forwarded). There would thus appear to be *four* centres of ossification, of which only three are to be found in the higher Mammals. The suggestion is that the fourth centre of ossification is at the basis of the parafibular sesamoid of the Marsupials, or the later *cyamella* and lateral *fabella*.

If the above view be correct that the Mammals with lateral sesamoids of the knee-joint are descended from a primitive mammalian form with a crested fibula, we should anticipate that still lower in the evolutionary scale, e.g. in the *Aves* or *Reptilia*, no lateral *fabella* or *cyamella* will be discoverable. It may be objected

that the present hypothesis provides no account of the origin of the mesial *fabella*. That is quite true, but it has appeared over and over again in the course of our investigation that, whatever the origin of the mesial *fabella*, that origin must be differentiated from that of the lateral *fabella*. Not only has the mesial come later in most orders and disappeared in the higher types much sooner than the lateral, but it is far less significant in size and far rarer as an anomaly. Further, while the lateral head of *gastrocnemius* in the earlier forms is attached not to the lateral condyle but to the head of the fibula, the mesial head retains its association with the internal condyle even in the Monotremes. It would demand much further study than we have been able at present to give to the mesial *fabella* to trace satisfactorily its origin and the source of its frequently imperfect condition. We would note, however, that if the lateral *fabella* and *cyamella* arise from a crested fibula, one might seek for a similar origin of the mesial *fabella* (and the reputed mesial *cyamella*?) in a like extension of the head of the tibia. Perhaps in this the cnemial crest of the tibia in certain birds—the *Tubinares* and still more the *Pygopodes*—might be borne in mind*. According to Shufeldt it has (like the fibular crest in Platypus) a separate centre of ossification and may be regarded as representing the olecranon of the ulna. Vicq d'Azyr as early as 1774, and he was followed by Owen, considered the patella itself to be homologous with the olecranon†. It is difficult to accept the view of the more modern writers that fibular or tibial crests are only extensions of the upper epiphysis of these bones if we find they have separate centres of ossification. Still less does it help us to be told that the patella is only a sesamoid, unless we hold the view that a sesamoid can be produced by some mechanical process when and wherever it is likely to be of service‡. At any rate if it can be, there seems no reason why we should stop at a sesamoid—the production of which would be at present as inexplicable as that of any new organ, and would require at least an acceptance of Lamarckism to ensure its perpetuation.

Besides the cnemial crest of birds as a vague suggestion§, there is a faint possibility of the transfer of a mesial posterior *lunula*, but we are unable to cite a

* See W. P. Pycraft, *Proc. Zool. Soc. London*, 1899, pp. 401 and 1037.

† "Mémoire sur les Rapports qui se trouvent entre les usages et la structure des quatre extrémités dans l'Homme et dans les Quadrupèdes," *Histoire [et Memoires] de l'Académie Royale des Sciences*, 1774, pp. 254—270, Paris, 1778. Vicq d'Azyr associates tibia with the ulna and fibula with the radius. He does not, however, claim the association of patella and olecranon to have originated with himself, although many later writers have attributed it to him,—"la rotule qui tient lieu d'olécrâne comme plusieurs anatomistes l'ont démontré" is how he writes on pp. 260—261.

‡ "I could imagine if further strength were required a sesamoid would be called into being"—as an anatomist once wrote to one of the present authors.

§ There are evidences that the cnemial crest is not a specialised development reached only in a small group of birds. Thus Shufeldt found it as a vestige—disappearing with growth in the young of *Centrocercus*, and states that in the present state of our knowledge this is incomprehensible, "for in the old and mature birds of any of the Grouse the epicnemial crest is never very prominently produced, nor is it in any of their near kin. As age advances this segment becomes thoroughly confluent with the tibia, and leaves no trace of its early existence." "Osteology of the North American Tetraonidae," *Bulletin of the U.S. Geological Survey*, Vol. VI. p. 344, 1881.

case of any fibres of the mesial head of *gastrocnemius* being attached to the semi-lunar. Such cases may, however, exist.

(21) *Birds.*

We have not examined a large number of birds' pelvic limbs, but we have not found in any of the cases we have examined nor in mounted skeletons either *fabellae* or *cyamella*. On the other hand we have found *lunulae*, for example, in *Syrnium stridula* (the Wood Owl) and *Dendrocygna* (the Tree Duck), the latter having in our specimen at least three posterior *lunulae*. The tibia of this bird also possesses a marked cnemial crest (see Plate XXII, Figs. 52, 53 and 54) apparently articulating with a small patella; this crest reaches to the level of the top of the patella. In Divers, Grebes, Albatrosses (see Plate XXII, Figs. 50 and 51), etc., this crest formed from an extension of the middle and internal crests of the tibia becomes a still more emphasised process and can even be as long as the femur itself*.

The homologue to the tibial crest appears to be the olecranon of the ulna†. Owen has noted in certain birds an ossiculum belonging to the ulna which he states is "essentially the separated olecranon of that bone. This detached sesamoid bone is found attached (like the patella of the knee-joint) to the capsular ligament and the tendons of the extensor muscles in many of the *Raptores* and in the Swifts. In the Penguins it is double‡." It would seem more justifiable to compare the cnemial crest of the tibia with the olecranon than the latter with the fibular process of the Monotremes, and not unreasonable to suppose that if the olecranon can give rise to sesamoids, a similar change can take place in the cnemial crest. In *Apteryx australis* there is a semi-cnemial crest surmounted by a cartilaginous patella. It is true that in some birds we know with marked cnemial crests the patella still exists, partly articulating with the femur and partly with the cnemial crest, but in *Colymbus* it is reduced to a "flake-like bone§." The patella, as we have remarked, has been associated with the olecranon by Vicq d'Azyr‖. Shufeldt, while associating the olecranon with the cnemial crest, looks upon the patella as not arising from its partial break-up, but treats the patella as a sesamoid.

We have carefully studied the evidence given by Shufeldt¶, but it seems to us quite as capable of interpretation in favour of the patella being originally the part or whole of the cnemial crest, as of both having quite distinct and separate origins. He first cites the great diversity of patellar forms in birds, from its absence in certain species, as *Haemotopus niger*, to the small double almost cartilaginous

* According to Owen its importance arises from the provision it affords for extensive attachments by way of insertions for the extensors of the tibia and by way of origin to the extensor of metatarsus. Thus the power of the back stroke of the foot is increased: see Todd's *Cyclopaedia of Anatomy and Physiology*, Vol. I. p. 287.

† Coues writes of *Colymbus torquatus* (*loc. cit.* on our p. 392) that the tibial crest exactly resembles the olecranon of the human species (p. 158).

‡ Owen's article *Aves* in Todd's *Cyclopaedia*, Vol. I. p. 286.

§ Of *Colymbus septentrionalis* (Red-throated Diver) we have examined two skeletons, but found no signs of a *patella*, although the semi-lunars were *in situ*.

‖ *Histoire de l'Académie Royale des Sciences*, Paris, 1774, p. 261.

¶ "Concerning some of the forms assumed by the Patella in Birds," *Proceedings U.S. National Museum*, Vol. VII. pp. 324—331, Washington, 1886.

patella of *Mergus servator*, to the flake-like patella of *Colymbus*, up to the huge quadrate patella of *Aptenodytes pennantii*. But he does not seem to lay any stress on the fact that where the cnemial crest is large as in *Colymbus* and *Fulmarus rodgersii* the patella is small, and where the patella is large as in *Podiceps* the cnemial crest is relatively small. Yet this result seems all important and easily interpretable on the assumption of the patella arising from a tibial crest. Lastly Shufeldt gives no explanation of the condition in the cormorant *Phalocrocorax bicristatus* of which he writes that it is almost unique; that on the face of it it almost looks as if a patella had developed of a size equal to that of the tibial crest and subsequently the two became thoroughly united and formed one large patella. This suggestion appears to us as an inversion of the real state of affairs. It also seems to be D'Arcy W. Thompson's view*. He treats the large patella of *Hesperornis* as homologous with the tibial crest of *Colymbus*. He actually speaks of this crest as a "co-ossified patella." He writes: "The existence of a small additional sesamoid in the knee-joint of *Colymbus* (Owen, *Comp. Anat.* II. p. 83) does not invalidate the homology here adopted of the long 'rotular process of the tibia' with the patella."

If the cnemial crest be identified with the patella, then it seems to us more reasonable to look upon the patella as a released tibial crest, than the tibial crest as a co-ossified patella†. It has not yet been suggested that the olecranon is a co-ossified sesamoid. Perhaps one of the chief difficulties in the idea of the tibial crest as a "co-ossified" patella is the large free patella of the cormorant; here the pro- and ecto-cnemial ridges of the tibia are carried right up the anterior face of the patella itself! (See Plate XXXVI, Fig. 93 *b*.) Why should an independent sesamoid have developed these ridges? Unless we accept the view that a sesamoid can always arise where it is a convenience, this leaves us still without an explanation of the origin of either the sesamoid or the patella. We have at any rate to bear in mind the very close association of patella and cnemial crest where they co-exist‡.

We note also that whereas a true patella, an orthosesamoid, exists in all placental mammals, it is absent in many Marsupials, it is not invariable in Birds and has disappeared when we get down to the Reptiles. Somewhere between the Reptiles and the branchpoint of the Birds a patella had to arise, somewhere between the Reptiles and the branchpoint of the Monotremes a fibular process had to arise§,

* "On the systematic position of *Hesperornis*," *Studies from the Museum of Zoology in University College, Dundee*, Vol. I. p. 108, Dundee, 1890.

† In both *Hesperornis* and *Ichthyornis* the tibia has a considerable epicnemial crest, in the former as well as the massive patella, in the latter no patella is figured or referred to; whether none has been found or it is assumed not to exist, we cannot say. See Marsh, "*Odontornithes*." *U. S. Geological Exploration of the 40th Parallel*, Vol. VII. Plate XXXIV, 1880.

‡ In the fossil *Hesperornis regalis* which we have examined the patella so far accords with Professor D'Arcy Thompson's view, that it has at least a superficial resemblance to a liberated cnemial crest. See also Plate XXXVI, Fig. 92.

§ An approach to a fibular crest occurs in some birds; thus in the *Cereopsis Goose* (see Plate XXI, Figs. 48 and 49), the head of the fibula rises above the proximal articulating surface of the tibia, and works in a groove of marked character on the posterior face of the lateral condyle: There is also a substantial cnemial crest.

and somewhere between that branchpoint and that of the Marsupials the fibular process had to be replaced by a parafibula, and lastly somewhere between the branchpoint of the Marsupials and the early placental mammals a *cyamella* and a lateral *fabella* had to replace the parafibula. We can of course take the view that a process will disappear without vestiges, if it ceases to be useful, and that a sesamoid will appear whenever and wherever it is likely to be useful*. But on philosophical grounds there is need of a connected account of the evolutionary origin and subsequent history of even the sesamoids of the knee-joint.

(22) *Reptiles.*

The chief feature of the knee-joint sesamoids of the Reptiles consists in the presence usually of a considerable number of *lunulae*. Parker and Haswell† speaking of the young condition of certain Marsupials state that there is an accessory element situated outside the fibula at its proximal end; "this apparently corresponds to a bone known as the parafibula which occurs in some *Lacertilia*." When the *cyamella* has left the popliteal groove and articulates with the head of the fibula, it becomes very difficult to say whether we are speaking of a *cyamella* or a parafibula. In our opinion the parafibula ultimately is divided into two elements; the upper portion becomes associated with the lateral head of *gastrocnemius* and is the lateral *fabella* of the placentals, the lower portion associated with *popliteus* is the *cyamella* proper. Hence when there is no *fabella* as in the Reptiles and Marsupials it seems a more reasonable course to term the single large sesamoid, which articulates with the head of the fibula and usually at the same time with the under surface of the lateral condyle, the parafibula. At the same time *popliteus* in most of these cases arises from the fibula, it may also be united with the semilunar, and it is not always easy to distinguish this form of the *cyamella*, a small parafibula, from a lateral anterior *lunula*. Our drawings, Plate XXIV, Figs. 64 *a* and 64 *b*, show a typical reptilian knee-joint, that of *Varanus niloticus* (Nile monitor). The anterior aspect shows the *cyamella* as a parafibula. The *popliteus* has a tendon in the popliteal groove of the femoral condyle, it is also attached to the superior posterior corner of the fibula, just internal to attachment of the external lateral ligament shown in the drawing. There are two anterior *lunulae*. The posterior aspect shows two posterior *lunulae*, a small mesial and a much larger lateral *lunula*‡. In *Varanus* the *cyamella* has become entirely anterior; it thus differs widely in position from the *cyamella* in higher mammalian forms. Cf. the Orang in our Plate XXVI, Figs. 68 and 69. This position of the *cyamella* is not universal in reptiles. Thus in *Trachysaurus rugosus*, the Australian stump-tailed lizard, the *cyamella* (see our Plate XXIV, Figs. 63 *a* and 63 *b*) is seen from *both* posterior and anterior aspects, and it is more easily confused with a *lunula* of the lateral semilunar. The knee-joint of this lizard has in addition two anterior *lunulae* and two posterior *lunulae*, of which the internal is larger than the external—exactly the opposite to what occur in *Varanus*.

* We have already indicated the difficulty of arguing as Gruber does the value of the knee-joint sesamoids to man in face of the fact that 90% of mankind do just as well without them.

† *Text-book of Zoology*, Vol. II. p. 524.

‡ In the specimen of *Varanus niloticus* mounted in the British Museum (Natural History) only the parafibula and the two anterior *lunulae* are visible.

Our third dissection was of a knee-joint of *Amphibolurus barbata*. Here we have no longer the parafibula, but posterior mesial and lateral *lunulae*, mesial anterior and lateral (or rather sagittal or trochlear) anterior *lunulae* (see our Plate XXIV, Figs. 65*a* and 65*b*). In this case the lateral posterior *lunula* is larger than the mesial, and there is a groove on the trochlear portion of the femur—exaggerated in the drawing to indicate its existence—which probably articulated with the sagittal *lunula*. In a very young specimen of *Iguana tuberculata* (*Lacertilia*) we were unable to find any parafibula; the posterior lateral *lunula* and the anterior sagittal *lunula* were present*. Mivart, who has studied the myology of *Iguana tuberculata*†, states that the outer head of *gastrocnemius* in this case arises from above the external condyle, that *plantaris* is very intimately connected with *gastrocnemius* and that *popliteus* arises from the *fibula*. He makes no mention of a parafibular sesamoid. In considering the absence of the parafibula this arrangement of the muscles should be borne in mind.

Specimens of *Agama stellio* and of a common lizard (*Lacerta vivipara*) dissected in our laboratory confirm the arrangement in *Varanus* of a parafibula with two anterior and two posterior *lunulae*.

In a specimen of *Spenodon punctata* examined no parafibula was found but only an anterior sagittal (mid-trochlear) *lunula*. This was confirmed from a mounted specimen of *Spenodon* seen at the South London Botanical Institute.

Mounted skeletons of the *Reptilia* do not as a rule afford much information, either when large on account of being over clean, or when small the knee-joints being often inaccessible even with the use of a lens to minute examination. They require dissection when moist. We may note the following observations made chiefly on specimens at the Royal College of Surgeons.

We examined three chameleons. In *Chamaeleo vulgaris*, we found a sesamoid at the top of the fibula and between this and external face of lateral condyle; there were no *lunulae* (or, of course, *fabellae*); the patella was ossified. In a specimen of *Chamaeleo verrucosus* there were cavities from which it looked as if parafibular sesamoids had fallen out. In a specimen of *Chamaeleo bifurcatus* nothing was distinguishable. In *Zonosaurus madagascarensis* there was an ossified patella, but no other visible knee-joint sesamoid. In *Eumeces algeriensis* there was a parafibula wedged between fibula and tibia; we also found a single *lunula*.

In *Corucia zebrata* there was a somewhat noteworthy system of knee-joint sesamoids. There was a sesamoid articulating with the head of the fibula and a second, directly connected by a ligament with the first, articulating with the head of the tibia; it would be easy to interpret either or both as cases of *lunulae*: see drawing, Plate XXXV, Fig. 91. There was a small ossified patella. Parafibular and paratibial sesamoids also occur in *Lacerta ocellata* together with an internal anterior *lunula*‡.

* In a mounted specimen of *I. tuberculata*, we again found no parafibular sesamoid, but the anterior *lunula* was rather lateral than sagittal.

† *Proc. Zool. Society of London*, 1867, p. 795.

‡ This lizard had, but on the left limb only, what looked like a possible lateral *fabella*.

In *Tiliqua scincoides*, *Heloderma suspectum* (the Gila monster), *Amieva surinamensis* and *Callopestes* sp. we were only able to note the parafibular sesamoid.

In *Amblyrhynchus cristatus* only lateral posterior and anterior *lunulae* were visible.

In *Physignathus lesueuri* there were all four *lunulae*. In *Uromastix spinipes* there was a small lateral and large mesial posterior *lunulae* and possibly very small lateral and mesial anterior *lunulae*.

In *Jacare punctata* we found what was possibly a mesial posterior *lunula*, but this was doubtful. In no mounted specimens of crocodiles or alligators of a large size have we found parafibular sesamoid or *lunulae**.

(23) *Amphibia*.

Among the Batrachians we dissected two species of frogs without discovering any sesamoids at all in the knee-joint. We examined a large number of skeletons with the same result. There were, however, two possible exceptions. In *Leptodactylus pentadactylus* we believe that a mesial posterior *lunula* may exist, and in *Docidophryne gigantea* with more doubt a lateral posterior *lunula*.

Finally we may note that we have dissected specimens of *Trionyx gangetica* and *Testudo graeca* and found no ossifications at all in the knee-joint, even the patella being cartilaginous.

It will thus be seen that the *Amphibia* and *Reptilia* present a great variety of sesamoidal conditions in the knee-joint. We may start from no ossifications at all, even the patella being cartilaginous; we may have a single anterior sagittal *lunula*†, or again two anterior and two posterior *lunulae*; or we may find in *Lacertilia* all four *lunulae* associated with a parafibula.

We have no real evidence of any lateral *fabella* in this order. The origin of this must be sought in the break-up of the parafibula into *cyamella* and lateral *fabella* in a stage between marsupial and primitive placental. One point appears to us clear, that the appearance of *lunulae* in the higher orders is a vestige of a reptilian ancestry, even in the case where *lunulae* appear in the Primates (from Lemuroids to Man). Further the appearances of *cyamella* and lateral *fabella* are vestiges (even in Man) of an ancestry of primitive ornithodelphian character which had a parafibula, or possibly a fibular crest.

The origin of the mesial *fabella* seems to us shrouded in far greater mystery; it is always of secondary importance, often merely cartilaginous, still more often absent; it is not a "Knotenpunkt" of important muscle attachments like the lateral *fabella*. The cnemial crest of the tibia is relatively rare and has been very rarely indeed

* In a mounted specimen of *Crocodilus americanus* we noted what looked like a very tiny external *fabella*!

† The possibility of an anterior sagittal *lunula* being the source of a patella is possibly worth bearing in mind.

recorded in reptilian forms*, and we do not know to what extent the inner head of *gastrocnemius* is associated with this crest†. The possibility of the cnemial crest having given rise to either or both patella and mesial *fabella* may indeed be borne in mind, but it is not of a high order of probability. One feels inclined to propound some general ontogenetic principle by which what occurs by evolutionary origin on one side of a member will be "sympathetically" (or cytologically?) shadowed on the other. Thus *lunulae*, *fabellae* or even condyles occur in pairs, and this is true also of muscular attachments and tuberosities. To propound such a principle would, however, be too venturesome‡, and all we can assert is that the mesial *fabella* has certainly a different origin to the lateral. It is later in arriving and earlier in degenerating and never has the anatomical importance of its lateral colleague.

(24) *Conclusions.*

The main point that seems to us to arise from this study of the sesamoids of the knee-joint is that no parts of the mammalian skeletal form, however apparently insignificant, are without exceeding interest if time can be devoted to their study. If we take the whole evolution of the sesamoids of the knee-joint from their entire absence in the Amphibia to their culminating point in the *Tupaiidae* and *Sciuridae* and to their entire disappearance in certain *Pteropodidae*, we are compelled to admit the feebleness of the argument for use value in any case but that of the patella. Even in the case of the patella we have to admit that the Amphibia, many reptiles, the Marsupials, and some birds seem to have got on remarkably well

* The occurrence of reptilian forms with even moderate tibial crests should be of a very exciting character. The only case we have so far been able to discover is that of *Ceratosaurus nasicornis* Marsh, a carnivorous Dinosaur of the American Jurassic. It is figured by Marsh and the tibia has a large cnemial crest. It shows in certain respects other avian resemblances as in the form of the pubis and in the co-ossifications of all the pelvic and of the metatarsal bones: see "The Dinosaurs of North America," by O. C. Marsh, *Sixteenth Annual Report of the U. S. Geological Survey*, Part I. pp. 156—162, and Plate XIV, 1896.

† E. Coues in his paper on "The Osteology of *Colymbus torquatus*, with notes on its Myology" (*Memoirs of the Boston Society of Natural History*, Vol. I. 1863, pp. 131—172) deals with muscles of the knee-joint. In this case the tibial crest is as long as the femur itself (2′): see our Plate XXXVI, Fig. 94, and the author makes no mention of an independent patella. Indeed he writes (p. 159): "Just at the border of the joint there is a very small projecting process of bone, which is generally regarded as the true analogue of the patella." Of the muscles he states that *Cruraeus* has a very extensive muscular attachment to the posterior face of the whole length of the tibial spine which projects above the knee-joint (p. 168). *Semimembranosus* is inserted by a broad membranous tendon into the crest of the tibia (p. 170). *Gastrocnemius* might, it seems, almost be described as having four heads. Part of the inner arises fleshy from the apex of the tibial spine and from the anterior face of the upper two-thirds of the bone, the other portion of the inner arises from the inner condyle. The outer head arises from the *linea aspera* for nearly two-thirds of its length and from the outer condyle; it also receives attachments from the *rectus femoris* (p. 171). But the *rectus femoris* instead of going to the patella as in man is inserted in the head of the fibula (p. 167). It would thus appear that the inner head arises from inner condyle and tibial crest, the outer head from outer condyle but is linked also to head of the fibula. The femur articulates with tibia and fibula; the articulation of inner condyle is tibial; that of outer condyle partly tibial, but most extensively fibular (p. 159). Without being conclusive this description is most suggestive for the possibility of the origin of the patella from the tibial crest and to a lesser extent of the origin of a mesial fabella from the same source.

‡ The tendency, however, to reach greater complexity by doubling may, perhaps, be traced in various organs, e.g. in the olfactory organ, or the evolutionary history of the ventricle of the heart.

without ossified patellae. If it be argued that this depends on a difference of mode of progression, a similar counter can hardly be made to the arguments adopted in favour of a use value for the *fabella*; these arguments are too strong, the *Anthropoidea* ought never to have lost their *fabellae*! If we take what appears to us the more rational view that all the sesamoids of the knee-joint are vestigial, and are debris of one or more structures originally of use value, we are bound to consider some such scheme as that of the accompanying diagram, and then the chief difficulty of linkage is seen to be at once apparent in the question as to how far the sesamoids in the knee-joint of any existing species are to be accounted for by acquisition or lapse. For example, the bear and the sheep possess the patella alone: are we to assume their ancestors have never possessed or have lost a complete system of knee-joint sesamoids? The frog and some species of bat possess no sesamoids at all of the knee-joint, but the former has never acquired them, and in the latter they have been suppressed in a long course of evolution. So it is with sheep and bear; the former belongs to the Ungulates who give no evidence of ever having developed a knee-joint sesamoid* beyond the patella, and the latter to the Carnivores with an elaborate evolutionary history of knee-joint sesamoids as complex as that of the Rodents or Insectivores. Again those who like Abel assume the Pinnipedia to have sprung from the *Ursidae*† seem to overlook the relationship of the latter to the full knee-joint sesamoid system of the Carnivores—especially to the *Canidae*.

Or, to go a stage further, we have so far not come across any Insectivores with patella only. Edentates of this character exist, but the Insectivore with only a patella has yet to be found. It may have existed, but the Insectivores seem to have sprung from a multi-sesamoidal form. Chiroptera and Rodentia can well have sprung from the Insectivores. To suggest as Kükenthal‡ that the Ungulates sprung from the Insectivores and the latter from Marsupials, is to neglect the fundamental contribution of the Marsupials to the knee-joint sesamoid system, namely, the *cyamella* and the lateral *fabella*. The parafibula is known in reptilian forms and the Edentates may have got their parafibula from these. But if the Ungulates sprung from the Marsupials it is astonishing that no cases even as anomalies of the *cyamella* in the Ungulates have been recorded.

We may note also a very special point of difficulty about the Marsupials themselves. An ossified patella is an extreme rarity. Are we to assume that the patella was lost in the Marsupials while *cyamella* and *fabellae* were developing? The ossified patella exists in the Monotremes, and this alone would differentiate them largely from the Marsupials. But if the patella was obtained in the reptilian stage and handed on to the Monotremes and certain birds, while it was lost in the Marsupials, it is hard to believe in a direct descent of Insectivores from Marsupials, at any rate it would point either to the *Peramelidae* or to some other Marsupial differing widely—as far as the knee-joint is concerned—from those hitherto known

* The lateral *fabella* of *Hyrax* we take to be evidence, not of a lost sesamoidal system in the Ungulates, but of a differentiated origin of *Hyrax*. We have not been able to confirm Meckel's and Gruber's statements as to the *Cervidae*.

† *Handwörterbuch der Naturwissenschaften*, Bd. VIII. S. 718, 1913. ‡ *Ibid.* Bd. VIII.

A study of either Carnivores or Primates indicates what a long course of evolution is involved in the loss of a sesamoidal system, and what a wide range of changing forms usually intervenes. The persistency of the same type of sesamoidal system for generations in the same species, and the occurrence of atavisms, such as *lunulae* in man, form strong evidence that the minor sesamoids of the knee-joint are as hereditary in character as the patella.

To assert that a sesamoid can be produced by "intensive stress" when and where it is required is to assert in its baldest and most futile form the inheritance of acquired characters. The most futile form because a parafibula, say, is not a merely formless mass, it is an organised link for a system of muscles, usually of a very elaborate kind. And if it could be acquired by "intensive stress" when and where needed, there would seem no reason to despair of any species producing a new bone of the skeleton, when and where needed. It is conceivable that a mutation could produce such a bone by an anomalous disturbance of the germ cells—but it would certainly not be due to "intensive stress" at the required point, and it would only be by the rarest chance that it would be "when and where needed." The whole mechanism of heredity is disregarded by such anatomical suggestions.

In forming the scheme of our diagram we wish to emphasise the fact that while there are factors of acquisition and suppression at work, we yet consider the primary factor to be heredity—the form of the sesamoidal system is inherited as the form of the teeth are inherited, and the change of form will in each case be a lengthy evolutionary process*.

In our diagram we have purposely refrained from providing any descent links, partly because there is such an immense amount of work still to be done on the sesamoid systems especially of reptilian and avian forms, and partly because to do so would be sure to give rise to the criticism that we had supposed one illustrative type to have sprung directly from a second illustrative type; whereas in fact both illustrative types may have been only *in cunabula* at the time of the link. The link lies between the sesamoidal systems, not necessarily between their present representatives.

We have really four origins to study:

(*a*) That of the patella; it may have arisen from the tibial crest or from an antero-sagittal *lunula*, or perhaps have had both origins. At present we have not come across any evidence of a tibial crest in the Reptilia except in the Dinosaurs.

(*b*) That (or better, perhaps, those) of the *lunulae*. We have no suggestion to make here. It needs a far greater study of the knee-joint of both Amphibia and Reptilia than we have been able to make.

* That in an early period of evolution the germ cells of various allied species were not so highly specialised and differentiated as in a later period, is a possibility that we have not seen emphasised, and which would, perhaps, render by hybridisation the earlier evolutionary processes quicker. Even now we have very vague ideas of the limits of possible hybridisation. How far are the Primates *inter se* fertile? The fox and dog are natural enemies, but the cross is a feasible one, and will provide a second generation. Much of importance for evolution might arise by crossing—if needful by artificial impregnation—forms apparently fairly diverse. Divergence of habits and temperament and somatic differences may have more to do with the problem of infertility than any factor of the germ cells themselves.

THE EVOLUTION OF THE ORTHOSESA

Acquisition ↓

No Sesamoids
(Most *Amphibia*)

No Sesamoids
(Some *Reptilia*)
Single Lunula
(Some *Amphibia*? e.g. *Leptodactylus pentadactylus*)

No Sesamoids
Lunulae
(Some *Reptilia*, e.g. *Amphibolurus barbata*)
Tibial Crest
(Some *Reptilia*?)
Fibular Crest
(Some *Reptilia*??)

Tibial Crest
Tibial Crest and Lunulae
(Some *Aves*)
Tibial Crest and Lunulae
(Some *Reptilia*?)
Parafibula and Lunula
(Some *Reptilia*, e.g. *Vara*

Tibial Crest
(Some *Aves*, e.g. Albatross)
Tibial Crest, Flake Patella and Lunulae
(Some *Aves*, e.g. *Colymbus*)
Patella, Parafibula and Lunulae
(Some *Reptilia*? e.g. *Corucia zebrata*)
P

Patella and Lunulae
(Some *Aves*)
Patella, Parafibula and Lunulae
?

Patella
(Some Edentates, *Dasypus*, *Chlamydophorus*)
Patella and Lunulae
?
Patella, Parafibula and Lunulae
(Some Edentates, *Megatherium*, *Bradypodidae*,

Patella
(Some Insectivores??)
Patella and Lunulae
(Some Insectivores, Myogale, perhaps)
Patella, Lunulae, L. Fabella
(Some Insectivores, e.g. *Talpa*, *Potamogale*)

Patella
(*Ungulata*, *Pinnipedia*)
Patella and L. Fabella
(*Hyracoidea*. ? Some *Cervidae*)
Patella, Lunulae, L. and M. Fabella
(Some Insectivores, e.g. *Erinaceus*, *Tupaia*, *Hylomys*)

Lapse ↓

Patella and Lunulae
(Some Rodents, e.g. *Castor fiber*)
Patella, Lunulae, L. and M. Fabellae
(Some Rodents, e.g. *Mus musculus*, *Fiber zibethicus*, *Dasyprocta*)

Patella, L. and M. Fabellae, Cyamella
(Some Carnivores, e.g. *Felis domestica*)
(Some

Patella, L. and M. Fabellae
(Some Carnivores, e.g. *Felis nebulosa*)
Patella and Cyamell
(Some *Anthropoidea*, Orang,

Patella and L. Fabella
(Some Carnivores, e.g. *Lutra*, *Gula vittatus*)
Patella
(Some *Anthropoidea*, Chimpanzee, and

Patella
(Some Carnivores, e.g. *Ursus*)

ORTHOSESAMOIDS OF THE KNEE-JOINT (INCLUDING THE PATELLA).

bular Crest
e *Reptilia* ??)

rafibula and Lunulae — Patella and Fibular Crest
Reptilia, e.g. *Varanus*) — (Monotremes)

Lunulae — Parafibula — Fibular Crest
ia zebrata) — ? — (Some Marsupials, e.g. occasionally Wombat)

Parafibula
(Some Marsupials; usually Wombat)

ula and Lunulae — L. Fabella and Cyamella
ium, *Bradypodidae*, *Manidae*) — (Some Marsupials, e.g. *Macropus*)

. Fabella — Patella, and L. Fabella — Patella and Cyamella — Patella, Cya
Talpa, *Potamogale*) — (Some Edentates? *Orycteropus*) — (Some Edentates, e.g. *Myrmecophaga*) — (Some Marsupials, e.g

M. Fabella — Patella, Cyamella, Lunulae — Cyamella and (or fused with) L. Fabella, M. Fabell
Tupaia, *Hylomys*) — (Some Insectivores, e.g. *Ericulus setosus*) — (asserted of *Halmaturus*)

Patella, Lunulae, Cyamella, L. and M. Fabellae — Patellae, Lunulae, Cyamella, L. and M. Fabellae
Dasyprocta) — (Some Lemuroids, e.g. Loris and Potto) — (Some Rodents, e.g. Rabbit, Squirrel)

Patella, Cyamella, L. and M. Fabellae — (S
(Some Lemuroids, e.g. *Chiromys*; *Cebidae*, and *Simiadae*)

Patella and Cyamella — Patella, L. and M. Fabellae — (Sc
nthropoidea, Orang, Gorilla) — (Certain *Hylobatinae*)

Patella — Patella and L. Fabellae
a, Chimpanzee, and *Hominidae*, Man) — (Certain *Hylobatinae*

Sc

To face p. 394

A).

Patella, Cyamella and (or fused with) L. Fabella
ne Marsupials, e.g. *Perameles*, bony kernel to patella in others)
Fabella, M. Fabella
urus)
and M. Fabellae
, Squirrel)

Patella, Cyamella, L. and M. Fabellae
(Probable ancestor of *Chiroptera*)

Patella, L. and M. Fabellae
(Some *Chiroptera*, e.g. *Cynopterus marginatus*)

Cyamella, L. and M. Fabellae
(Some *Chiroptera*, e.g. certain species *Vespertilio*)

L. and M. Fabellae
(Some *Chiroptera*, e.g. *Vespertilio murinus*)

L. Fabella
(Some *Chiroptera*, e.g. *Phyllostoma*)

No Sesamoids
Some *Chiroptera*, e.g. certain species *Pteropus*)

(*c*) That of *cyamella* and lateral *fabella*. We are fairly convinced that this lies in the break-up of the parafibula and that the parafibula was the result of the separation of the fibular crest with its attached muscular system. The parafibula exists in certain reptiles. The discovery of a reptilian fibular crest would be of the greatest importance. Of course there may be again a double origin corresponding to the parafibula of the Reptiles and the fibular crest of the Monotremes.

(*d*) That of the appearance at a relatively late stage of evolution (and the early disappearance) of the mesial *fabella*. We have no real light to throw on this point; it needs a very special study. The fact that the mesial *fabella* is never specific without the lateral, while the lateral is in early as well as in late evolutionary stages specific without the mesial, suggests that the explanation may be cytological. At the same time we have to remember that while the lateral head of *gastrocnemius* had originally a fibular or at least a parafibular attachment, the evolutionary history of the inner head seems to point to no such marked change in its attachments*.

The object of this memoir has been to suggest problems to those better equipped for studying them than the present authors, rather than to present solutions. They believe that the least important elements of the skeletal form are each able to contribute important suggestions for the general attack on the enigma of evolution and that all vestiges are of special value in the assault. They might even be content with the outcome of this paper if it led anatomical text-book writers in the future to drop once for all the Galen tradition and assert on the contrary that it is needful that all sesamoids should be discussed.

As final conclusion may we not cite the words used by Pittard in 1849 as appropriate to our own views: "But after all, taking into consideration all the facts related above, and many others that have presented themselves to us in the course of this enquiry, we cannot but believe that some higher law than that of adaptation concurs in determining the presence, if not the size, of even these little bones†."

We have to acknowledge a grant from the Dixon Fund for Scientific Investigation allotted by the Senate of the University of London towards the cost of the illustrations of this memoir.

We wish further to thank Sir Arthur Keith for the invariable kindness he has shown us in freely placing the immense stores of valuable material in his charge at our disposal. For aid in a great variety of enquiries we have to thank very cordially Dr E. C. Derry of Cairo, Mr W. R. Sherrin of the South London Botanical Institute, Dr W. C. Mackenzie of the Australian Institute of Anatomical Research, Melbourne, and Dr Julia Bell of the Galton Laboratory. We have also to acknowledge the very helpful assistance of Miss I. McLearn, especially in the drawings of the knee-joints of birds.

* It may have in certain cases an insertion to the tibial crest: see the second footnote on p. 392.

† Article on Sesamoid Bones, Todd's *Cyclopaedia of Anatomy and Physiology*, p. 543.

DESCRIPTION OF PLATES XIX—XXXVII*.

Plate XIX, Fig. 40. Femur of Sea Otter (*Enhydra lutris*), showing the marked development of the lateral *fabella*; the sesamoid appears by its action to have largely eroded the popliteal surface of the femur, and the periosteum of the bone in its neighbourhood has disappeared.

Fig. 41. Fibula, Tibia and Femur of Beaver (*Castor fiber*). The development of the fibular head at A suggests how a second sesamoid might well originate in the tendon of the muscle (*Peroneus longus*?) attached to this projection. The femur exhibits a marked third trochanter.

Plate XX, Fig. 42. Drawing of musculature of knee-joint of *Ornithorhynchus*. We see the lateral head of *M. gastrocnemius* attached solely to the fibular crest; the mesial head of *gastrocnemius* is still attached to the mesial condyle.

Fig. 43 *a*, anterior, and Fig. 43 *b*, lateral aspect of the femur, tibia and fibula of *Ornithorhynchus*. The fibula shows the noteworthy crest, a compressed process with its own centre of ossification. There are no sesamoids except patella. Cf. Fig. 47 for the corresponding bones of *Echidna*.

Fig. 44. Drawing of the posterior aspect of the left knee-joint of *Felis domestica*, showing the positions of the external (*e.f.*) and internal (*i.f.*) *fabellae*, the *cyamella* (*cy.*) and the facet on which the latter rests. Owing to the knee-joint being slightly extended the *cyamella* has been lifted off its articular surface on the head of the fibula†.

Fig. 45. Drawing of postero-lateral aspect of the left knee-joint of *Hyrax* showing the external *fabella* (*e.f.*) above the external condyle, but the tendon of *gastrocnemius* has shrunk in drying, so that the *fabella* has been lifted off the condyle. There is a slight depression on the internal condyle, which might suggest an internal *fabella*, but so far none has been found.

Plate XXI, Fig. 46. *Colymbus glacialis*, a drawing after Owen's figure in his article *Aves* in Todd's *Cyclopaedia*. Note the remarkable cnemial crest of the tibia. Cf. especially all the figures on Plate XXXVI.

Fig. 47. Lateral aspect of pelvic limb of *Echidna* to be compared with the knee-joint of *Ornithorhynchus* in Figs. 43 *a* and 43 *b*. Drawing after a part of the skeleton in a plate of Cuvier's *Recherches sur les ossemens fossiles*, Paris, 1834–6 (Tome II, Atlas Plate 214).

Fig. 48. Drawing of lateral, Fig. 49, drawing of mesial, aspect of knee-joint of left limb of Cereopsis Goose, showing cnemial crest of tibia and also a semi-fibular crest which works in a groove on the articular surface of the lateral condyle. Cf. Plate XXXVI.

Plate XXII, Figs. 50 and 51. Drawings from lateral and mesial aspects of the left knee-joint of the Albatross, showing the remarkable cnemial crest of the tibia.

Figs. 52, 53 and 54. Drawings of anterior, lateral and posterior aspects of knee-joint of *Dendrocygna*, the Tree Duck. There is a cnemial crest (*cn. crest*), internal (*i.p.lu.*) and external (*e.p.lu.*) posterior *lunulae* and also what appear to be two or three very small sesamoids wedged in between fibula and tibia (*sesamoids*). Cf. Plate XXXVI.

Plate XXIII, Fig. 55 *a*. Tamandua Anteater (*Tamandua tetradactyla*); photograph of posterior aspect of right limb showing the large *cyamella* (*c.*) still in the popliteal tendon.

Fig. 55 *b*. Common Squirrel (*Sciurus vulgaris*), photograph of posterior aspect of right limb, showing internal and external *lunulae* (*i.l.* and *e.l.*) and *cyamella* (*c.*). Both *fabellae* also exist in the squirrel.

Figs. 56 *a* and 56 *b*. Common Hedgehog (*Erinaceus europaeus*). The former photograph of posterior aspect of right limb showing internal (*i.f.*) and external (*e.f.*) *fabellae*; the latter photograph of mesial aspect of left limb with the femur doubled right back to show anterior external (*a.e.l.*) and anterior internal (*a.i.l.*) *lunulae*. No *cyamella* was found.

Fig. 57 *a*. Photograph of the right pelvic limb of *Hydromys chrysogaster*, the Golden-bellied Rat, from the posterior aspect showing internal (*i.f.*) and external (*e.f.*) *fabellae*.

* The drawings are as a rule natural size.

† It must be remembered that in nearly all these drawings the sesamoid bones are still contained in the tendons in which they occur. Consequently no attempt could be made to show the actual size of the sesamoids.

Fig. 57 *b*. Photograph of the posterior aspect of the right pelvic limb of *Fiber zibethicus*, the Musquash, showing internal (*i.f.*) and external (*e.f.*) *fabellae* together with a posterior external *lunula* (*p.e.l.*). Compare Figs. 61 *a* and 61 *b*.

Plate XXIV, Fig. 58. Drawing of posterior aspect of left pelvic limb of *Lepus cuniculus*, the Common Rabbit, showing large external (*e.f.*) and smaller internal (*i.f.*) *fabellae* and *cyamella* (*cy.*). The rabbit has usually (?) two anterior *lunulae* as well, but not in this specimen.

Fig. 59. Drawing of lateral aspect of left pelvic limb of *Potamogale velox* from West Africa, showing the external *fabella* (*e.f.*). No sign of ossification in popliteal tendon. Fibula fused with tibia at distal end.

Fig. 60 *a*. Drawing of posterior aspect of left knee-joint of *Parazerus jacksoni*, African Squirrel, showing external (*e.f.*) and internal (*i.f.*) *fabellae*, together with *cyamella* (*cy.*).

Fig. 60 *b* is the anterior view of the same knee-joint showing an internal anterior *lunula* (*a.i.lu.*).

Fig. 61 *a* and Fig. 61 *b*. Drawings respectively of the posterior and anterior aspects of the right knee-joint of *Fiber zibethicus*, the Musquash or Musk Rat. The former shows the internal (*i.f.*) and external (*e.f.*) *fabellae* with a posterior external *lunula*; the latter besides the two *fabellae* shows external (*e.a.lu.*) and internal (*i.a.lu.*) anterior *lunulae*, but no *cyamella* was present. Cf. Plate XXIII, Fig. 57 *b*.

Fig. 62 *a* and Fig. 62 *b* provide drawings of the posterior and postero-lateral aspects of right pelvic limb of *Hylomys suillus dorsalis*, a Tree Shrew from Borneo. Only internal (*i.f.*) and external (*e.f.*) *fabellae* were found.

Fig. 63 *a* and Fig. 63 *b*. Drawings of the anterior and posterior aspects of the knee-joint of *Trachysaurus rugosus*, the Australian stump-tailed lizard. The former shows at *a'* the lateral anterior *lunula*, and at *b'* the mesial anterior *lunula*; the latter gives *a* the mesial and *b* the lateral posterior *lunula*, and the *cyamella* as indicated. The *cyamella* is on the *anterior* corner of the fibula, but is seen in both posterior and anterior aspects. The four *lunulae* all lie between tibia and femur, the *cyamella* between fibula and femur.

Fig. 64 *a* and Fig. 64 *b*. Drawings of the posterior and anterior aspects of the right knee-joint of *Varanus niloticus*, the Nile Monitor. The former shows internal (*p.i.lu.*) and external (*p.e.lu.*) posterior *lunulae* and the latter external (*a.e.lu.*) and internal (*a.i.lu.*) anterior *lunulae* as well as a large *cyamella* or better *parafibula* (*paraf.*) on the head of the fibula. This parafibula has tendon in the popliteal groove and is also attached to the head of the fibula; it lies just internal to the attachment of the external lateral ligament as shown in the drawing (Fig. 64 *b*). It will be seen that the *cyamella* is entirely anterior and this position is in partial agreement with that of *Trachysaurus rugosus*, which, however, is also visible from posterior as well as anterior aspect. Cf. Figs. 63 *a* and 63 *b*.

Fig. 65 *a* and Fig. 65 *b*. Drawings of the anterior and posterior aspects of the left knee-joint of *Amphibolurus barbata*. The former shows two anterior *lunulae*, a small internal one (*a.i.lu.*) and a larger external (*a.e.lu.*) or rather mid-trochlear *lunula*. The latter shows an external (*p.e.lu.*) and an internal (*p.i.lu.*) posterior *lunula*; both are on top of the tibia and the external is the larger. There is no trace of a parafibula.

Plate XXV, Fig. 66. Photograph of the right knee-joint of *Megatherium cuvieri* or the Fossil Giant Sloth, in the Royal College of Surgeons' Museum. The photograph shows a sesamoid on the top of the fibula, which has usually been described as a lateral *fabella*. It lies close to the popliteal groove and there is little doubt that it is a *cyamella* (*cy.*) or rather parafibula. On the top of the tibia is a second sesamoid which we have not hitherto seen referred to. It is undoubtedly an external anterior *lunula* (*a.e.lu.*). It lies externally from the base of the large *patella* (*pa.*).

Fig. 67. Photograph of the left knee-joint of *Manis aurita*, the Chinese Pangolin. The knee-joint has been opened out for the purpose of photography, but the *cyamella* (*cy.*), usually described as a *fabella*, appears to have been mounted in the right position. When the knee is flexed the sesamoid does not lie on the top of the external condyle, but on the top of the fibula; it is a parafibula, articulating with both fibula and the *posterior* face of the external condyle.

Plate XXVI, Fig. 68. Drawing of the posterior aspect of left knee-joint of *Simia satyrus*, the Orang-ûtan, showing the *cyamella* (*cy.*) in position in the popliteal tendon. Inset, drawings of three *cyamellae* from three Orang-ûtans' knee-joints dissected in the Biometric Laboratory, posterior and anterior aspects,

showing range of sizes from *b* to *c* and *a*. *a* is from Orang-ûtan figured, *b* and *c* are from a second specimen, left and right joints respectively. No other sesamoids were found.

Fig. 69 is a drawing of the same knee-joint as in Fig. 68 with the popliteal tendon containing the *cyamella lateralis* (*cy.*) turned back so as to disclose the facet (*f. for cy.*) on which it articulates.

Plate XXVII, Fig. 70 *a* and Fig. 70 *b*. Drawings from posterior and lateral aspects respectively of the left knee-joint of *Lemur varia*, showing the external (*e.f.*) and internal (*i.f.*) *fabellae*, the *cyamella* (*cy.*) and an anterior external *lunula* (*a.e.lu.*) in position. The *cyamella* is seen in close relation to the head of the *fibula* and one notes how with flexed limb the external *fabella* will come into contact with it.

Fig. 71 *a*. Drawing of posterior aspect of right knee-joint of *Mycetis palliatus*, showing internal *fabella* (*i.f.*), external *fabella* (*e.f.*) and *cyamella* (*cy.*).

Fig. 71 *b*. Sesamoids from the joint in Fig. 71 *a*. *a* is posterior aspect, *b* anterior aspect of *cyamella*, *c* and *d* are respectively *fabella mesialis*, and *fabella lateralis* in posterior aspects.

Fig. 72. Drawing of distal extremity of right femur of a Gibbon (*Hylobates lar*) which possessed only an internal sesamoid (*i.f.*) and no ossification in the popliteal tendon (*p.t.*) which was "horny" but not ossified.

Fig. 73. Drawing of postero-lateral aspect of left knee-joint of *Lemur melanocephalus* showing external (*e.f.*) and internal (*i.f.*) *fabellae*, the *cyamella* (*cy.*) enclosed in popliteal tendon, and an anterior external *lunula*.

Plate XXVIII, Fig. 74 *a*. Drawing of posterior aspect of left knee-joint of *Chiromys*, the Aye-Aye. The two branches of *M. gastrocnemius* (g_1 and g_2) have been deflected to show the external (*e.f.*) and the internal (*i.f.*) *fabellae* lying beneath; below is the popliteal tendon (*p.t.*) with the *cyamella* lying on the head of the fibula (*fib.h.*) and articulating with the tibia. *po.* is the popliteal muscle.

Fig. 74 *b*. Drawing of the lateral aspect of the left knee-joint of *Chiromys* after Owen (*Trans. Zool. Soc. London*, Vol. v, 1866, Plate XIX) to indicate how in flexure of the knee the external *fabella* (*e.f.*) comes in contact with the *cyamella* (*cy.*).

Fig. 75. Drawing in posterior aspect of left knee-joint of *Lemur catta*, showing external (*e.f.*) and internal (*i.f.*) *fabellae*, the *cyamella* (*cy.*) and an anterior external *lunula* (*a.e.lu.*).

Fig. 76. Drawing of the lateral aspect of the left knee-joint of a dry specimen of *Perodicticus potto*, Bosman's Potto, showing the position of the anterior external *lunula* (*a.e.lu.*) and the *cyamella* (*cy.*) on the head of the fibula. The popliteal tendon sends a branch down to the head of the fibula (see our p. 378). The *fabellae* had disappeared from this specimen, but it is clear that, in this flexed position of the limb, the lateral *fabella* must rest on the proximal surface of the *cyamella*.

Fig. 77. Drawing of the lateral aspect of the left knee-joint of *Indris brevicaudata*, the Indri, showing the popliteal tendon (*p.t.*) containing the *cyamella* (*cy.*) and the positions of external (*e.f.*) and internal (*i.f.*) *fabellae*. The intimate relations of external *fabella* and *cyamella* on complete flexure are again indicated.

Plate XXIX, Fig. 78. Right, photograph of right pelvic limb of *Phascolomys wombatus*, the Wombat. This specimen shows the parafibula absolutely fused to the fibula so as to show one continuous bone with very small indication of position of union.

Left, photograph of right pelvic limb of *Sarcophilus ursinus*, the Tasmanian Devil; the parafibula is united to the fibula, but its limits are more clearly defined.

Fig. 79. Photograph of left pelvic limb of Wombat, the two portions of the parafibula, which is really free of the fibula, are shown in position on the head of the fibula, the femur is tied back to show up the parafibula which in the usual flexed position would be in contact or almost in contact with the shaft of the femur.

Plate XXX, Fig. 80. Photograph of the left pelvic limb of *Phalangista vulpina*, the Wolfine Phalanger, showing the free parafibula (*paraf.*) mounted on the head of the fibula and its relation to the femur when the limb is flexed. The two portions of the parafibula, which ultimately become the lateral *fabella* and the *cyamella*, are easily distinguished.

Plate XXXI, Fig. 81. Drawing of the posterior aspect of the right knee-joint of the Wombat with muscles attached. Description chiefly on figure itself. The external *gastrocnemius* has been cut across and the proximal portion deflected to show parafibular sesamoid lying in its fibres, and the surface of the fibula with which the sesamoid articulates. The internal *gastrocnemius* takes origin from nearly the

whole of the popliteal surface, while the external *gastrocnemius* originates from the parafibular sesamoid. The drawing is also intended to show the tendinous fibres running from *gastrocnemius externus* to the lateral border of the patella; a few, cut through in drawing in order to expose the patella, run to the lateral face of the external condyle. The popliteal tendon can also be seen taking origin in the popliteal groove of the external condyle. It passes across the head of the fibula, its internal border being connected with the external semilunar cartilage, while the external fibres run down to the muscle fibres of the popliteus. This latter arises from the upper third of the posterior surface of the fibula. The double nature of *gastrocnemius externus* suggests that the deeper portion is *soleus*, although Macalister ("Myology of the Wombat and the Tasmanian Devil," *Annals of Natural History*, 1870, p. 19 (offprint)) suggests that *soleus* has a fibular origin.

Plate XXXII, Fig. 82 *a* and Fig. 82 *b*. Drawings of the posterior aspect of the left pelvic limb of *Phascolarctus cinereus*, the Koala, with the muscles attached. Fig. 82 *a* shows the popliteal muscle and the external *gastrocnemius* with the parafibular sesamoid covered by its fibres. Fig. 82 *b* shows *gastrocnemius* cut and turned back, so that the facet of the sesamoid which articulates with the fibula is exhibited and the position on which it rests on the fibular head is seen. This sketch is also intended to illustrate the following relations. The fleshy portion of *popliteus* is attached to fibula, and popliteal tendon runs from the anterior proximal corner of fibula to popliteal groove on external condyle. Also tendinous fibres from the fleshy portion on fibula cross the posterior surface of the latter and mingle with the tendinous fibres of *gastrocnemius* which run from the sesamoid to the external condyle. (See also Fig. 86.)

Plate XXXIII, Fig. 83 *a* and Fig. 83 *b*. *Macropus* sp. The portion of the knee-joint of a kangaroo (species ?) here drawn was most kindly sent to us by Dr W. C. Mackenzie of the Australian Institute of Anatomical Research. It consists only of the lateral branch of *gastrocnemius* with its sesamoid and the *cyamella*, together with the external semilunar cartilage attached. It was thus not possible to draw these in their natural position. It should not be impossible, however, to understand the relations to other structures if the reader will compare these drawings with those of other *Marsupialia*. The three important components are the external *fabella*, the cartilaginous *cyamella*, with surface which articulates with the fibular head, and the external semilunar cartilage. These are drawn in two aspects, the interesting point being that in the kangaroo the lower part of the parafibula, still articulating with the fibular head on the distal face and with the external *fabella* on the proximal has become purely cartilaginous.

Fig. 84. Drawing of the posterior aspect of the left knee-joint of *Didelphys virginiana*, the Virginian Opossum. Here *a* is the popliteal tendon from groove on external condyle, *b* is the *cyamella* in the popliteal tendon as it passes over fibular head, *c* and *d* are branches of popliteal muscle running from tendon attached to *cyamella*, *e* is the external *fabella* in tendon of *gastrocnemius*, *g* and *g'* are the cut ends of *gastrocnemius*. The *gastrocnemius* has been cut just below its insertion on top of the external condyle to enable it to be pulled aside and display the *fabella*. When the leg is fully flexed the external *fabella* lies on the external condyle, and the *cyamella* is in direct contact with the lower surface of the condyle and the *fabella*.

Fig. 85. Drawing of the lateral aspect of the left knee-joint of *Phalangista vulpina*, the Wolfine Phalanger. *a* is the parafibular sesamoid resting on the head of the fibula, *b* is the popliteal tendon. If the reader will compare this drawing with Fig. 78, he will grasp how parafibula and fibula united give a single complete bone. It is not possible to term the sesamoid of Phalangista merely a *fabella*.

Fig. 86. A further drawing of the posterior aspect of the right knee-joint of *Phascolarctus cinereus* after removal of the muscles. *a* is the tendon of *gastrocnemius externus* attached to femur above external condyle, *b* is the cut end of *gastrocnemius externus*, *c* is the fibular "collateral" ligament, *d* is the popliteal tendon running to head of fibula, the lower portion of popliteal tendon has been removed. From a specimen kindly sent by Dr W. Colin Mackenzie.

Plate XXXIV, Fig. 87. Femur of *Macropus bennettii*, the Wallaby, with parafibular sesamoid attached above the external condyle.

Fig. 88. *Perameles lagotis*, the Rabbit-eared Perameles, with parafibular sesamoid attached above the external condyle.

Figs. 87 and 88 are photographs of museum specimens and we hold that they place the sesamoid in an erroneous position, leading it to be confused with a lateral *fabella*.

Fig. 89 *a* and Fig. 89 *b*. Photographs of anterior and lateral aspects of a moist preparation of the right knee-joint of *Sarcophilus ursinus*, the Tasmanian Devil, from the Museum of the Royal College of Surgeons, showing the parafibular sesamoid in what we believe to be the true position, i.e. mounted on the head of the fibula with a ligament connecting its summit with the lateral face of the femur above the external condyle.

Plate XXXV, Fig. 90. Drawing of the lateral aspect of the right knee-joint of *Halmaturus sp.*, the Wallaby, to indicate the position of the parafibular sesamoid on the head of the fibula.

Fig. 91. Drawing of the postero-lateral aspect of the left limb of *Corucia zebrata* to show a noteworthy sesamoid arrangement. *a* is a parafibular sesamoid united by a ligament to *b* a paratibial sesamoid, which might be a *lunula*. Drawing by Mr S. Steward.

Plate XXXVI. Drawings of the knee-joints of Birds. Cf. Plate XXI, Figs. 46, 48, 49, and Plate XXII, Figs. 50, 51, 52, and 53.

Fig. 92. Drawing of lateral aspect of left pelvic limb **A** of *Hesperornis regalis*, after Marsh. Note the giant patella—longer than femur—and the condyles of the femur enclosed by patella; tibia and fibula.

Fig. 93 *a* and Fig. 93 *b* give the mesial (B_1) and lateral (B_2) aspects of the right and left knee-joints of the Cormorant; the patella is of relatively smaller size than that of *Hesperornis regalis*, but clearly built on the same lines; the continuity of the ridges *aa'* and *bb'*,—the pro- and ecto-cnemial ridges of the tibia carried right up the lateral face of the patella,—must be emphasised, as in favour of released cnemial crest rather than a co-ossified patella.

Fig. 94. Drawing of the lateral aspect of the left pelvic limb **C** of *Colymbus torquatus* after Coues. There is a giant cnemial crest. A comparison of **A** and **C** suggests either that (i) the cnemial crest is a co-ossified patella or (ii) the patella a released cnemial crest; independent origin of both is hard to accept. B_2 is certainly an argument in favour of (ii).

Plate XXXVII, Fig. 95 and Fig. 96. Drawings of sections respectively of the fibular crest of a young and an embryo platypus. In the former case ossification is complete but the arrangement of the Haversian canals at the top of the plate, i.e. the proximal end of the fibular crest, seems far from incompatible with a separate centre of ossification. In the latter there appear signs of ossification starting towards the summit of the crest in an area widely removed from the centres of ossification of the shaft or of the fibular head.

THE INCOMPLETE MOMENTS OF A NORMAL SOLID.

By A. RITCHIE-SCOTT, D.Sc.

1. *Introductory.*

One of the chief problems of mathematical statistics is the representation as closely as possible of a series of observations by a mathematical formula, or speaking graphically, by fitting a geometrically determinate curve or surface to the observations, these being represented by some kind of space unit. The formula, curve or surface constitutes a résumé of the known facts and gives that conception of them which is the scientific law underlying their relations. The very conception, however, of the statistical method involves a classification of the phenomena into groups, which assumes identity for the purposes of classification and ignores the infinitely fine gradation within the group. We have therefore to deal, not with the actual measurements, but with groups of measurements labelled with some representative value or class mark and to these groups we must fit our formula, curve or surface.

In the process of fitting, the method of most universal application is the method of moments which consists of expressing "the area and moments of the curve or surface for the given range of observations in terms [of the real constants of the theoretical curve] and equating these to the like quantities for the observations" (K. Pearson, "On the systematic fitting of curves," *Biometrika*, Vol. I, 1902, p. 270).

In the early history of mathematical statistics Gauss fitted the normal curve to observations by means of zero, first and second moments, i.e. by means of the sum, mean and standard deviation of the observations. The method of least squares is for any method of polynomial fitting a method of moments in which high moments may have to be used.

It was soon discovered, however, that the incomplete moments of the normal curve are important particularly in regard to plural partial correlation and the fitting of incomplete curves, while the development of the ideas of multiple correlation and variation brought into view the need of multiple moments and multiple product moments which are still further required in the evaluation of the probable error of multiple correlation coefficients.

With multiple variates we have the same problem as with the single variate, viz. the reconstruction of a population from a portion of it, and for this purpose incomplete moments and product moments are essential. Further, the theory of plural partial multiple correlation of observations classed in broad categories depends entirely on a knowledge of these incomplete product moments.

It is therefore from several points of view very desirable to obtain algebraical expression for these incomplete moments, and the present paper is an attempt to deal with the problem.

In the course of the analysis it was found necessary to employ functional operators which lead to the development of functions of the form of

$$x^n - \frac{n.n-1}{2}x^{n-2} + \frac{n.n-1.n-2.n-3}{2.4}x^{n-4}\ldots.$$

These in a somewhat different form have been termed tetrachoric functions and their values tabled (*Tables for Statisticians and Biometricians*, Cambridge, 1914). In order to avoid confusion with these tabled functions and also to suggest their connection with multiple celled tables I have tentatively called the functions developed in the present paper, polychoric functions. The incomplete multiple product moment is found to be expressible in a form of multiple polychoric function which is itself reducible to a function of single polychoric functions.

2. *Notation.*

As the integrals in the following paper are very complicated unless an abbreviated notation is used, it will be necessary to resume some of the well-known formulae connected with the normal equation in order to be intelligible and to avoid confusion.

Consider the surface represented by the normal equation

$$Nz'(x', y') = \frac{N}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}}e^{-\frac{x'^2/\sigma_x^2 + y'^2/\sigma_y^2 - 2\cdot x'y'/\sigma_x\sigma_y}{2(1-r^2)}} \quad \ldots\ldots\ldots\ldots(1).$$

If we write $\frac{x'}{\sigma_x} = x$ and $\frac{y'}{\sigma_y} = y$ and

$$Nz(x, y) = \frac{N}{2\pi\sqrt{1-r^2}}e^{\frac{x^2+y^2-2rxy}{2(1-r^2)}} \quad \ldots\ldots\ldots\ldots(2),$$

then

$$z'(x', y') = \frac{z(x, y)}{\sigma_x\sigma_y} \quad \ldots\ldots\ldots\ldots(3).$$

If now the surface be cut by the plane $x' = h'$ then the area of the face of the section will be

$$N\int_{-\infty}^{+\infty} z'(h', y')\,dy' = \frac{Ne^{-h'^2/2\sigma_x^2}}{\sqrt{2\pi}\,\sigma_x} = \frac{NH}{\sigma_x} \quad \ldots\ldots\ldots\ldots(4).$$

If another plane be drawn at $y' = k'$ it will in a similar manner expose a surface whose area is

$$N\int_{-\infty}^{+\infty} z'(x', k')\,dx' = \frac{Ne^{-k'^2/2\sigma_y^2}}{\sqrt{2\pi}\,\sigma_y} = \frac{NK}{\sigma_y} \quad \ldots\ldots\ldots\ldots(5).$$

The two planes will divide the volume enclosed by the normal surface into four quadrants a, b, c, and d as in the usual tetrachoric scheme

a	b
c	d

in which the x's go from $-\infty$ to $+\infty$ in the direction of a to b and y's similarly from a to c. The a quadrant will be considered as the leading or standard quadrant

and where it is necessary to specify the limits of integration they will be those of this quadrant. Its volume will be denoted by m with, where necessary, a distinguishing suffix.

Hence we may write

$$m = N\int_{-\infty}^{h'}\int_{-\infty}^{k'} z'(x', y')\,dx'dy' = N\int_{-\infty}^{h}\int_{-\infty}^{k} z(x_1, y_1)\,dxdy \quad \ldots\ldots(6).$$

The area of the face bounding quadrant a parallel to the y axis is

$$N\int_{-\infty}^{k'} z'(h', y')\,dy' = \frac{Ne^{-h'^2/2\sigma_x^2}}{\sqrt{2\pi}\sigma_x}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{k-rh}{\sqrt{1-r^2}}} e^{-y^2/2}\,dy = \frac{NH}{\sigma_x}A \ldots\ldots (7).$$

Since the whole area of the section is $\frac{NH}{\sigma_x}$, A is the fraction which the portion bounding the quadrant a is of the whole section. It has no dimensions and is independent of σ_x and σ_y.

Similarly the face bounding the a quadrant parallel to the x axis is

$$N\int_{-\infty}^{h'} z'(x', k')\,dx' = \frac{Ne^{-k'^2/2\sigma_y^2}}{\sqrt{2\pi}\sigma_y}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{h-rk}{\sqrt{1-r^2}}} e^{-x^2/2}\,dx = \frac{NK}{\sigma_y}B \quad \ldots(8).$$

A and B may be taken from Sheppard's Tables when h, k and r are known.

Writing $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}x^2}\,dx = \mathscr{E}(x),$$

which is the function tabled, we have

$$A = \mathscr{E}\left(\frac{k-rh}{\sqrt{1-r^2}}\right) \text{ and } B = \mathscr{E}\left(\frac{h-rk}{\sqrt{1-r^2}}\right) \quad \ldots\ldots\ldots\ldots(9).$$

Lastly we have the ordinate at the intersection of the planes

$$Nz'(h', k') = \frac{Nz(h, k)}{\sigma_x\sigma_y} = \frac{N\chi}{\sigma_x\sigma_y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

Since
$$\chi = \frac{1}{2\pi\sqrt{1-r^2}}e^{-\frac{h^2+k^2-2rhk}{2(1-r^2)}}$$
$$= \frac{1}{\sqrt{1-r^2}}\cdot\frac{e^{-\frac{1}{2}h^2}}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{(k-rh)^2}{2(1-r^2)}}$$
$$= \frac{1}{\sqrt{1-r^2}}\cdot\frac{e^{-\frac{1}{2}k^2}}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{(h-rk)^2}{2(1-r^2)}} \quad \ldots\ldots\ldots\ldots(11),$$

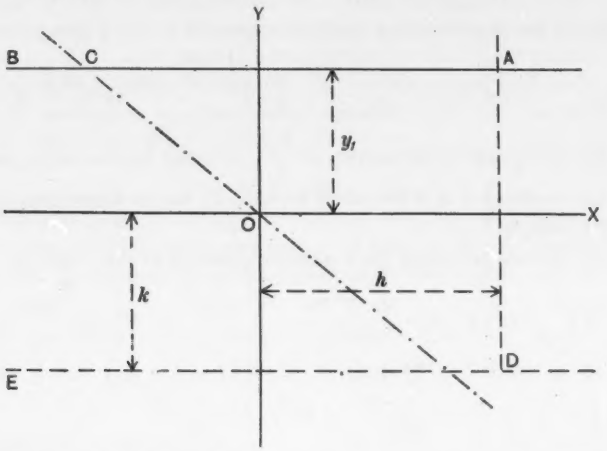
we may write

$$\chi = \frac{1}{\sqrt{1-r^2}}\cdot E(h)\cdot E\left(\frac{k-rh}{\sqrt{1-r^2}}\right)$$
$$= \frac{1}{\sqrt{1-r^2}}\cdot E(k)\cdot E\left(\frac{h-rk}{\sqrt{1-r^2}}\right) \quad \ldots\ldots\ldots\ldots(12),$$

where $E(x)$ is the tabled function $\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$. Clearly $E(h) = H$, and $E(k) = K$.

3. *Centroid of a normal quadrant found by geometry.*

In this and the following section I shall show how the lower moments may be found by direct methods and also how these methods lead to results of such complexity with high moments that a symbolic method is necessary.



Let OX, OY be the co-ordinate axes through the mean of a normal solid and let AD, DE be the projections of the bounding planes of a quadrant. AB is the projection of a plane cutting the solid perpendicularly to the Y axis and at a distance $y = y_1$. The equation of the curve in which it cuts the surface will be

$$Nz = N \frac{e^{-\frac{1}{2}y_1^2}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}\sqrt{1-r^2}} \cdot e^{-\frac{1}{2}\left(\frac{x-ry_1}{\sqrt{1-r^2}}\right)^2} \quad \text{.....(13),}$$

i.e. a curve whose mean is at $x = ry_1$, whose standard deviation is $\sqrt{1-r^2}$ and whose area is NK. Considering B to be at an infinite distance the area of BA is

$$N\int_{-\infty}^{h} z(x, y_1)\, dx = \alpha \text{ say,} \quad \text{.....(14),}$$

and since the centroid of a segment of a normal curve

$$= \frac{\text{ordinate}}{\text{area}} \times (\text{stand. dev.})^2 \quad \text{.....(15),}$$

writing the centroid of BA as $\bar{x}_{y_1}$ we have

$$\bar{x}_{y_1} - ry_1 = -\frac{(1-r^2).\, z(h, y_1)}{\alpha} \quad \text{.....(16).}$$

Therefore the x moment of the section AB may be written

$$\bar{x}_{y_1}\alpha = ry_1\alpha - (1-r^2).\, z(h, y_1) \quad \text{.....(17).}$$

Now let the surface BA move from $y = -\infty$ to $y = k_1$ then $\bar{x}_{y_1}\alpha$ will generate the x moment of the quadrant, $y_1\alpha$ will generate the y moment and $z(h, y_1)$ will generate the area of the surface exposed by the plane AD. Writing the centroids of the quadrant $\bar{\bar{x}}$, $\bar{\bar{y}}$

$$\alpha\bar{\bar{x}} = r\alpha\bar{\bar{y}} - (1-r^2)\int_{-\infty}^{k} z(h, y)\, dy \quad \text{.....(18).}$$

A similar discussion of the y moment will give the equation

$$\alpha\bar{\bar{y}} = r\alpha\bar{\bar{x}} - (1-r^2)\int_{-\infty}^{h} z(x, k)\, dx \quad \text{.....(19).}$$

Both the integrals are known and the centroids may be found as the solution of a pair of simultaneous equations.

4. *Another method of finding the moments.*

The method of § 2 may be expressed analytically and extended to higher moments. For example second moments may be found as follows:

$$\left.\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= -\frac{1}{(1-r^2)^2}\{x^2 - 2rxy + r^2y^2 - (1-r^2)\}\, z(x, y) \\ \frac{\partial^2 z}{\partial x \partial y} &= +\frac{1}{(1-r^2)^2}\{r(1-r^2) - [rx^2 - (1+r^2)xy + ry^2]\}\, z(x, y) \\ \frac{\partial^2 z}{\partial y^2} &= -\frac{1}{(1-r^2)^2}\{r^2x^2 - 2rxy + y^2 - (1-r^2)\}\, z(x, y) \end{aligned}\right\} \quad \text{...(20).}$$

Hence

$$\left.\begin{aligned} (x^2 - 2rxy + r^2y^2)\, z(x, y) &= (1-r^2)^2 \frac{\partial^2 z}{\partial x^2} + (1-r^2)\, z(x, y) \\ \{rx^2 - (1+r^2)xy + ry^2\}\, z(x, y) &= -(1-r^2)^2 \frac{\partial^2 z}{\partial x \partial y} + r(1-r^2)\, z(x, y) \\ (r^2x^2 - 2rxy + y^2)\, z(x, y) &= (1-r^2)^2 \frac{\partial^2 z}{\partial y^2} + (1-r^2)\, z(x, y) \end{aligned}\right\} \quad \text{...(21).}$$

Multiplying both sides by N and integrating, and writing

$$\left.\begin{aligned} m \,.\, \bar{m}_{20} &= N\iint x^2 \,.\, z(x, y)\, dx dy \\ m \,.\, \bar{m}_{11} &= N\iint xy \,.\, z(x, y)\, dx dy \\ &\text{etc.} \end{aligned}\right\} \quad \text{.....(22),}$$

we get

$$\left.\begin{aligned} \bar{m}_{20} - 2r\bar{m}_{11} + r^2\bar{m}_{02} &= \frac{N}{m}(1-r^2)^2 \iint \frac{\partial^2 z}{\partial x^2}\, dx dy + (1-r^2) \\ r\bar{m}_{20} - (1+r^2)\,\bar{m}_{11} + r\bar{m}_{02} &= -\frac{N}{m}(1-r^2)^2 \iint \frac{\partial^2 z}{\partial x \partial y}\, dx dy + r(1-r^2) \\ r^2\,\bar{m}_{20} - 2r\bar{m}_{11} + \bar{m}_{02} &= +\frac{N}{m}(1-r^2)^2 \iint \frac{\partial^2 z}{\partial y^2}\, dx dy + (1-r^2) \end{aligned}\right\} \quad \text{...(23).}$$

The integrals are known and the $\overline{m}$'s can be found by solving the equations. With high moments the equations become cumbrous and it is difficult to follow the relations between the moments. A more powerful method has therefore been adopted.

5. *Moments of a* $\left(\frac{1}{2^n}\right)^{ant}$ *of an n-fold normal correlation surface by means of functional operators.*

The rationale of the process to be developed in this paper will be best understood by considering at once the general case. The details for a normal surface in two variables will be worked out in a later section.

Consider the equation to a normal correlation surface

$$Nz(x_1, x_2, \dots x_n) = \frac{N}{(2\pi)^{\frac{n}{2}}\sqrt{\Delta}} e^{-\frac{\Sigma\Delta_{11}x_1^2 + 2\Sigma\Delta_{12}x_1x_2}{2\Delta}} \quad \dots\dots\dots\dots(24),$$

where as usual

$$\Delta = \begin{vmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \dots & r_{nn} \end{vmatrix} \quad \dots\dots\dots\dots\dots\dots\dots\dots(25),$$

and Δ_{11}, Δ_{12} are the co-factors of r_{11}, r_{12} ... and $r_{11} = r_{22} = \dots = 1$, and $r_{12} = r_{21}$ etc.

The general problem is to find the l_1, l_2, l_3 ... l_nth product moment of the $\left(\frac{1}{2^n}\right)^{ant}$ of the form, that is to reduce the integral

$$m \cdot \overline{m}_{l_1 l_2 \dots l_n} = N \int_{-\infty}^{h_1} \int_{-\infty}^{h_2} \dots \int_{-\infty}^{h_n} x_1^{l_1} x_2^{l_2} \dots x_n^{l_n} z(x_1, x_2 \dots x_n)\, dx_1 dx_2 \dots dx_n$$
$$\dots\dots(26).$$

Differentiating partially by x_1, x_2, ...

$$\left.\begin{aligned} \frac{\partial z}{\partial x_1} &= -\frac{1}{\Delta}(\Delta_{11}x_1 + \Delta_{12}x_2 + \dots)z \\ \frac{\partial z}{\partial x_2} &= -\frac{1}{\Delta}(\Delta_{12}x_1 + \Delta_{22}x_2 + \dots)z \\ &\dots\dots\dots\dots\dots\dots \\ \frac{\partial z}{\partial x_s} &= -\frac{1}{\Delta}(\Delta_{1s}x_1 + \Delta_{2s}x_2 + \dots)z \\ &\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \quad \dots\dots\dots\dots(27).$$

Hence remembering that Δ_{st} is the co-factor of r_{st} we have

$$\left.\begin{aligned} \left(r_{11}\frac{\partial z}{\partial x_1} + r_{12}\frac{\partial z}{\partial x_2} + \dots r_{1n}\frac{\partial z}{\partial x_n}\right) &= -\frac{1}{\Delta}(\Delta x_1)z = -x_1 z \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \left(r_{s1}\frac{\partial z}{\partial x_1} + r_{s2}\frac{\partial z}{\partial x_2} + \dots r_{sn}\frac{\partial z}{\partial x_n}\right) &= -\frac{1}{\Delta}(\Delta x_s)z = -x_s z \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \quad \dots\dots\dots(28).$$

Now representing the left side of the above equations by means of operators such as

$$-\left(r_{s1}\frac{\partial}{\partial x_1}+r_{s2}\frac{\partial}{\partial x_2}+\ldots\right)=\phi_s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29),$$

we have

$$\phi_1(z)=x_1.z;\ \ \phi_2(z)=x_2.z,\ \ldots\ \phi_s(z)=x_s.z \quad \ldots\ldots\ldots\ldots\ldots(30).$$

Further

$$\phi_1(x_1)=-r_{11}=-1;\ \ \phi_1(x_2)=-r_{12};\ \ \phi_2(x_1)=-r_{21};\ \ \phi_2(x_2)=-r_{22}=-1$$

and generally

$$\phi_s(x_{s'})=\phi_{s'}(x_s)=-r_{ss'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31).$$

Since these operators are weighted sums of differential coefficients they are distributive, commutative and iterative.

Hence

$$\left.\begin{aligned}
\phi_1^2(z)&=\phi_1(\phi_1(z))\\
&=\phi_1(x_1z)\\
&=\phi_1(x).z-x_1.\phi_1(z)\\
&=-r_{11}z+x_1^2z\\
\phi_1^3(z)&=\phi_1(\phi_1^2(z))\\
&=\phi_1(-r_{11}z+x_1^2z)\\
&=-r_{11}\phi_1(z)+2x_1.\phi_1(x).z+x_1^2\phi(z)\\
&=-r_{11}x_1z-2r_{11}x_1z+x_1^3z\\
&=-3r_{11}x_1z+x_1^3z\\
&\text{etc..}\\
\phi_1\phi_2(z)&=\phi_1(x_2z)\\
&=\phi_1(x_2).z+x_2\phi_1(z)\\
&=-r_{12}z+x_2x_1z
\end{aligned}\right\}\ \ldots\ldots\ldots\ldots(32).$$

In this manner any function $\phi_1^{l_1}\phi_2^{l_2}\ldots(z)$ may be expanded into a series of terms each of which involves the r's, the x's and z. That is we may write

$$\Sigma r_{ss'}r_{s''s'''}\ldots x_1^{p_1}x_2^{p_2}\ldots z=\phi_1^{l_1}\phi_2^{l_2}\ldots(z) \quad \ldots\ldots\ldots\ldots\ldots(33).$$

Now if both sides be integrated with respect to all the variables we shall have a sum of mixed moments on the left side and on the right side the integral

$$\int_{-\infty}^{h_1}\int_{-\infty}^{h_2}\ldots\int_{-\infty}^{h_n}\phi_1^{l_1}\phi_2^{l_2}\ldots\phi_n^{l_n}(z)\,dx_1dx_2\ldots dx_n \quad \ldots\ldots\ldots\ldots(34),$$

on the reduction of which the solution depends.

Consider the numerator of the index of e

$$\Sigma\Delta_{ss}x_s^2+2\Sigma\Delta_{st}x_sx_t=f(x_1x_2\ldots x_n) \quad \ldots\ldots\ldots\ldots\ldots(35).$$

Write

$$\begin{aligned}
x_1&=y_1+r_{11}h_1,\\
x_2&=y_2+r_{12}h_1,\\
&\ldots\ldots\ldots\ldots\\
x_s&=y_s+r_{1s}h_1,
\end{aligned}$$

then
$$f(x_1x_2\dots x_n)=f(y_1+r_{11}h_1,\ y_2+r_{12}h_1,\ \dots\ y_n+r_{1n}h_1)$$
$$= f(y_1,\ y_2,\dots y_n)$$
$$+\frac{h_1}{1!}\left(r_{11}\frac{\partial}{\partial y_1}+r_{12}\frac{\partial}{\partial y_2}+\dots\right)f(y_1,\ y_2\dots y_n)$$
$$+\frac{h_1^2}{2!}\left(r_{11}\frac{\partial}{\partial y_1}+r_{12}\frac{\partial}{\partial y_2}+\dots\right)^2 f(y_1,\ y_2\dots y_n)$$
$$+\dots.$$

But
$$\left(r_{11}\frac{\partial}{\partial y_1}+r_{12}\frac{\partial}{\partial y_2}+\dots\right)f(y_1,\ y_2\dots y_n)$$
$$= 2(\Delta_{11}y_1+\Delta_{12}y_2+\dots)r_{11}+2(\Delta_{12}y_1+\Delta_{22}y_2+\dots)r_{12}$$
$$+\dots$$
$$= 2(r_{11}\Delta_{11}+r_{12}\Delta_{12}+\dots)y_1+2(r_{11}\Delta_{12}+r_{12}\Delta_{22}+\dots)y_2$$
$$+\dots$$
$$= 2\Delta y_1,$$

and
$$\left(r_{11}\frac{\partial}{\partial y_1}+r_{12}\frac{\partial}{\partial y_2}+\dots\right)^2 f=\left(r_{11}\frac{\partial}{\partial y_1}+r_{12}\frac{\partial}{\partial y_2}+\dots\right).2\Delta y_1=2\Delta$$
and the remaining terms vanish.

Hence
$$f(y_1+r_{11}h_1,\ y_2+r_{12}h_1+\dots)=f(y_1,\ y_2\dots y_n)+2h\Delta y_1+\Delta h_1^2 \quad\dots\dots(36).$$

Put $y_1=0$ and
$$f(h_1,\ x_2,\ x_3\dots x_n)=f(0,\ x_2-r_{12}h_1,\ x_3-r_{13}h_1+\dots)+\Delta h_1^2$$
$$\text{say}=f(x_2',\ x_3'\dots x_n')+\Delta h_1^2\dots\dots\dots\dots\dots\dots\dots\dots(37)$$

Hence we may write
$$Nz(h_1x_2x_3\dots x_n)=\frac{N}{(2\pi)^{\frac{n}{2}}\sqrt{\Delta}}\,e^{-\frac{\Sigma\Delta_{ss'}x_s'^2+2\Delta_{s1}x_s'x_i'+\Delta h_1^2}{2\Delta}} \quad\dots\dots\dots(38)$$

Omitting the first variate from the Σ summation,
$$=\frac{e^{-h_1^2/2}}{\sqrt{2\pi}}\cdot\frac{N}{(2\pi)^{\frac{n-1}{2}}\sqrt{\Delta}}\cdot e^{-\frac{\Sigma\Delta_{ss'}x_s'^2+2\Delta_{s1}x_s'x_i'}{2\Delta}}$$

Hence.
$$\int_{-\infty}^{h_1}\int_{-\infty}^{h_2}\dots\int_{-\infty}^{h_n}\phi_1^{l_1}\phi_2^{l_2}\dots\phi_n^{l_n}z\,dx_1dx_2\dots dx_n$$
$$=\int_{-\infty}^{h_1}\int_{-\infty}^{h_2}\dots\int_{-\infty}^{h_n}\phi_1\phi_1^{l_1-1}\phi_2^{l_2}\dots\phi_n^{l_n}z\,dx_1dx_2\dots dx_n$$
$$=-\int_{-\infty}^{h_1}\int_{-\infty}^{h_2}\dots\phi_1^{l_1-1}\phi_2^{l_2}\dots\left(r_{11}\frac{\partial}{\partial x_1}+r_{12}\frac{\partial}{\partial x_2}+\dots\right)z\,dx_1dx_2\dots dx_n$$
$$=-\sum_1^s r_{1s}\int_{-\infty}^{h_1}\dots\left[\int_{-\infty}^{h_s}\right]\dots\int_{-\infty}^{h_n}\phi_1^{l_1-1}\phi_2^{l_2}\dots z(x_1,\ x_2\dots h_s\dots x_n)\,dx_1\dots[dx_s]\dots dx_n$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(39),$$

in which the expressions in brackets disappear,
$$=-\frac{e^{-h_s^2/2}}{\sqrt{2\pi}}\times\text{an integral of order } n-1.$$

Any ϕ integral may therefore be reduced one degree and since we can evaluate $\int_{-\infty}^{h} z(x, h)\,dx$ the formal solution of the problem is complete.

6. *Polychoric Functions.*

I shall now consider in detail the case for two variables, but before doing so it will be necessary to establish certain ancillary formulae which will greatly simplify both the process of integration and the computation when the formulae are used in numerical work. The first of these formulae are closely related to the well-known tetrachoric functions which have been tabulated in *Tables for Statisticians and Biometricians* (Cambridge, 1914), and to emphasise this relation I have ventured to call them polychoric functions. They are however already known as parabolic cylinder functions* (see Whittaker and Watson, *Modern Analysis*, p. 341, 2nd edition, Cambridge, 1915). Only the properties germane to the present issue will be dealt with here.

From the definition of $z'(x', y')$ we have

$$\left.\begin{aligned} \frac{\partial}{\partial x'} z'(x', y') &= -\frac{1}{1-r^2}\frac{1}{\sigma_x}\left(\frac{x'}{\sigma_x} - r\frac{y'}{\sigma_y}\right) z' \\ \frac{\partial}{\partial y'} z'(x', y') &= -\frac{1}{1-r^2}\frac{1}{\sigma_y}\left(\frac{y'}{\sigma_y} - r\frac{x'}{\sigma_x}\right) z' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (40),$$

and we may write

$$\left.\begin{aligned} \phi'(z') &= -\left(\sigma_x \frac{\partial}{\partial x'} + r\sigma_y \frac{\partial}{\partial y'}\right)(z') = \frac{x'}{\sigma_x} z' \\ \psi'(z') &= -\left(r\sigma_x \frac{\partial}{\partial x'} + \sigma_y \frac{\partial}{\partial y'}\right)(z') = \frac{y'}{\sigma_y} z' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (41),$$

where ϕ' and ψ' are functional operators.

In most cases there will be gain in clearness and no loss in generality in writing x for $\frac{x'}{\sigma_x}$ and y for $\frac{y'}{\sigma_y}$ and when desirable the following contracted forms will be used:

$$\left.\begin{aligned} \phi(z) &= -\left(\frac{\partial}{\partial x} + r\frac{\partial}{\partial y}\right)(z) = x \,.\, z \\ \psi(z) &= -\left(r\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(z) = y \,.\, z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (42).$$

Since

$$\left.\begin{aligned} \phi(z) &= x \,.\, z, \quad \psi(z) = y \,.\, z \\ \phi(x) &= -1, \quad \psi(x) = -r \\ \phi(y) &= -r, \quad \psi(y) = -1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (43),$$

[* For the history of the subject and applications of the functions: see Pearson, K., *Phil. Trans.* Vol. 195, A, pp. 1–47, 1900; Whittaker, E. T., *Proc. Lond. Math. Soc.* Vol. xxxv. pp. 417–427, 1903; Pearson, K., "A Mathematical Theory of Random Migration," *Drapers' Research Memoirs*, Biometric Series, III., Camb. Univ. Press, 1906; and Cunningham, E., *Proc. R. Soc.* Vol. 81, A, pp. 310–331, 1908. ED.]

then
$$\begin{aligned}\phi^2(z) &= \phi\{\phi(z)\}\\ &= \phi(x\,.\,z)\\ &= \phi(x)\,.\,z + x\,.\,\phi(z)\\ &= -z + x\,.\,xz\\ &= (x^2-1)\,z\end{aligned}$$

and any combination of operators may be developed in the same way.

Assume $\phi^n(z) = x\,.\,\phi^{n-1}(z) - (n-1)\,\phi^{n-2}(z)$ .....(44).

Then
$$\begin{aligned}\phi^{n+1}(z) &= \phi\{x\,.\,\phi^{n-1}(z)\} - (n-1)\,\phi\{\phi^{n-2}(z)\}\\ &= \phi(x)\,.\,\phi^{n-1}(z) + x\,.\,\phi^n(z) - (n-1)\,\phi^{n-1}(z)\\ &= x\phi^n(z) - n\phi^{n-1}(n) \quad \text{.....(45),}\end{aligned}$$

which is of the same form as (48). But the relation is true for $n = 1, 2$, and hence is generally true. The solution of this functional equation is

$$\phi^n(z) = \left(x^n - \frac{n\,.\,n-1}{2}\,x^{n-2} + \frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{2\,.\,4}\,x^{n-4} - \ldots\right) z \quad \text{...(46).}$$

Writing the factor

$$x^n - \frac{n\,.\,n-1}{2}\,x^{n-2} + \frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{2\,.\,4}\,x^{n-4} - \ldots = T_n(x) \quad \text{.....(47),}$$

we may write $\phi^n(z) = T_n(x)\,.\,z$,

and $T_n(x)$ I have called the polychoric function of x of the nth order. A more convenient method of writing it is

$$\frac{T_n(x)}{n!} = \frac{x^n}{n!} - \frac{x^{n-2}}{2\,(n-2)!} + \frac{x^{n-4}}{2^2\,.\,2!\,(n-4)!} - \ldots = \Sigma(-1)^s \frac{x^{n-2s}}{2^s\,.\,s!\,(n-2s)!} \quad \text{...(48).}$$

In a similar way it may be easily shown that

$$\bar{\phi}^n(\zeta) = \left\{x^n + \frac{n\,.\,n-1}{2}\,x^{n-2} + \frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{2\,.\,4}\,x^{n-4} + \ldots\right\}\zeta \quad \text{...(49),}$$

where $\bar{\phi} = \left(\dfrac{d}{dx} + r\,\dfrac{d}{dy}\right)$ and $\zeta = \dfrac{1}{2\pi\sqrt{1-r^2}}\,e^{+\frac{x^2+y^2-2rxy}{2(1-r^2)}}$,

and it will be found useful to write $x^n + \dfrac{n\,.\,n-1}{2}\,x^{n-2} + \ldots = \mathfrak{H}_n(x)$. The following property of these functions is sufficiently interesting to be noted here. Presuming the argument x throughout

$$\left.\begin{aligned}&\frac{T_n}{n!} + p\,\frac{T_{n-2}}{2\,(n-2)!} + p^2\,\frac{T_{n-4}}{2^2\,.\,2!\,(n-4)!} + \ldots\\ (a) \quad &= \frac{x^n}{n!} - \frac{x^{n-2}}{2\,(n-2)!} + \frac{x^{n-4}}{2^2\,.\,2!\,(n-4)!} - \ldots\\ (b) \quad &+ \frac{p}{2}\left(\frac{x^{n-2}}{(n-2)!} - \frac{x^{n-4}}{2\,(n-4)!} + \frac{x^{n-6}}{2^2\,.\,2!\,(n-6)!} - \ldots\right)\\ (c) \quad &+ \frac{p^2}{2^2\,.\,2!}\left(\frac{x^{n-4}}{(n-4)!} - \frac{x^{n-6}}{2\,(n-6)!} + \frac{x^{n-8}}{2^2\,.\,2!\,(n-8)!} - \ldots\right)\\ &+ \ldots\ldots\end{aligned}\right\} \quad \text{...(50).}$$

The $s+1$th term of the (a) line is $(-1)^s \frac{x^{n-2s}}{2^s . s!(n-2s)!}$

sth " (b) " $(-1)^{s-1} \frac{x^{n-2s}}{2^{s-1}(s-1)!(n-2s)!} \cdot \frac{p}{2}$

$s-1$th " (c) " $(-1)^{s-2} \frac{x^{n-2s}}{2^{s-2}(s-2)!(n-2s)!} \cdot \frac{p^2}{2^2 . 2!}$ } (51).

Hence the $s+1$th term of the whole expansion is

$$(-1)^s \frac{x^{n-2s}}{2^s . s!(n-2s)!} \left\{1 - \frac{s}{1!}p + \frac{s . s-1}{2!}p^2 - \frac{s . s-1 . s-2}{3!}p^3 + \ldots\right\}$$

$$= (-1)^s \frac{x^{n-2s}}{2^s . s!(n-2s)!}(1-p)^s = \frac{x^{n-2s}}{2^s . s!(n-2s)!}(p-1)^s \ldots\ldots (52).$$

Hence

$$\frac{T_n}{n!} + p\frac{T_{n-2}}{2(n-2)!} + p^2\frac{T_{n-4}}{2^2 . 2!(n-4)!} + \ldots$$

$$= \frac{x^n}{n!} - (1-p)\frac{x^{n-2}}{2(n-2)!} + (1-p)^2\frac{x^{n-4}}{2^2 . 2!(n-4)!} - \ldots \quad \ldots (53).$$

Putting $p = 1$

$$\frac{x^n}{n!} = \frac{T_n}{n!} + \frac{T_{n-2}}{2(n-2)!} + \frac{T_{n-4}}{2^2 . 2!(n-4)!} + \ldots \quad \ldots\ldots\ldots\ldots (54)$$

and $$x^n = T_n + \frac{n . n-1}{2}T_{n-2} + \frac{n . n-1 . n-2 . n-3}{2 . 4}T_{n-4} + \ldots \quad \ldots\ldots (55),$$

which may be symbolically written

$$x^n = \mathfrak{A}_n \{T(x)\} \quad \ldots\ldots\ldots\ldots (56),$$

if we apply suffixes to the T's on the right side instead of indices to an argument. In a similar manner it may be shown that

$$x^n = \mathfrak{A}_n - \frac{n . n-1}{2}\mathfrak{A}_{n-2} + \frac{n . n-1 . n-2 . n-3}{2 . 4}\mathfrak{A}_{n-4} - \ldots \quad \ldots (57),$$

which may be symbolically written

$$x^n = T_n \{\mathfrak{A}(x)\} \quad \ldots\ldots\ldots\ldots (58),$$

and finally $$\mathfrak{A}_n \{T(x)\} = T_n \{\mathfrak{A}(x)\} \ldots\ldots\ldots\ldots (59).$$

From its mode of formation it is obvious that the functional equation of $T_n(x)$ is

$$T_n(x) = x . T_{n-1}(x) - (n-1)T_{n-2}(x) \quad \ldots\ldots\ldots\ldots (60).$$

If we make $n = -1$ and expand we get

$$T_{-1}(x) = \frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \ldots \quad \ldots\ldots\ldots\ldots (61).$$

But $$\int_{-\infty}^{x} e^{-\frac{1}{2}x^2} dx = -e^{-\frac{1}{2}x^2}\left(\frac{1}{x} - \frac{1}{x^3} + \ldots\right) = -T_{-1}(x) . e^{-\frac{1}{2}x^2}.$$

Hence $$T_{-1}(x) = -\frac{\int_{-\infty}^{x} e^{-\frac{1}{2}x^2} dx}{e^{-\frac{1}{2}x^2}} = -\frac{\mathscr{E}(x)}{E(x)} \quad \ldots\ldots\ldots\ldots (62).$$

From the operational point of view

$$\int_{-\infty}^{x} e^{-\frac{1}{2}x^2}\,dx = \int_{-\infty}^{x} \phi\phi^{-1}\left(e^{-\frac{1}{2}x^2}\right).dx = -\,\phi^{-1}\,.\,e^{-\frac{1}{2}x^2} = -\,T_{-1}(x)\,.\,e^{-\frac{1}{2}x^2}$$

as before.

I have hitherto considered the polychoric function with respect to one variable only. If now we consider two variables the query arises where there is a corresponding function such that

$$\phi^p\psi^q(z) = T_{p,q}(x, y)\,.\,z(x, y) \quad \text{.....(63),}$$

where ψ stands to y in the same relation that ϕ stands to x. We have

$$\phi\left(\frac{x^n}{n!}\right) = -\frac{x^{n-1}}{(n-1)!}; \quad \psi\left(\frac{x^n}{n!}\right) = -\,r\,\frac{x^{n-1}}{(n-1)!}.$$

Hence
$$\phi^s\left(\frac{x^n}{n!}\right) = (-1)^s\frac{x^{n-s}}{(n-s)!}; \quad \psi^s\left(\frac{x^n}{n!}\right) = (-r)^s\frac{x^{n-s}}{(n-s)!},$$

and
$$\phi^s\left(\frac{T_n(x)}{n!}\right) = \phi^s\left\{\frac{x^n}{n!} - \frac{x^{n-2}}{2\,(n-2)!} + \frac{x^{n-4}}{2^2\,.\,2\,!\,(n-4)!} - \ldots\right\}$$

$$= (-1)^s\left\{\frac{x^{n-s}}{(n-s)!} - \frac{x^{n-s-2}}{2\,.\,(n-s-2)!} + \frac{x^{n-s-4}}{2^2\,.\,2\,!\,(n-s-4)!} - \ldots\right\}$$

$$= (-1)^s\frac{T_{n-s}(x)}{(n-s)!} \quad \text{.....(64),}$$

and similarly

$$\psi^s\left(\frac{T_n(x)}{n!}\right) = (-r)^s\frac{T_{n-s}(x)}{(n-s)!} \quad \text{.....(65),}$$

with reciprocal relations between ϕ, ψ and $T_n(y)$. Hence

$$\begin{aligned}
\frac{\psi^q\phi^p}{q!\,p!}(z) &= \frac{\psi^q}{q!}\left(\frac{T_p(x)}{p!}\,.\,z\right) \\
&= \frac{\psi^q}{q!}\left(\frac{T_p(x)}{p!}\right).z \\
&+ \frac{\psi^{q-1}}{(q-1)!}\cdot\left(\frac{T_p(x)}{p!}\right)\cdot\frac{\psi}{1!}(z) \\
&+ \frac{\psi^{q-2}}{(q-2)!}\cdot\left(\frac{T_p(x)}{p!}\right)\cdot\frac{\psi^2}{2!}(z) \\
&+ \ldots\ldots \\
&= \left\{\frac{(-r)^q}{q!}\cdot\frac{T_{p-q}(x)}{(p-q)!} + \frac{(-r)^{q-1}}{(q-1)!}\cdot\frac{T_{p-q+1}(x)}{(p-q+1)!}\cdot\frac{T_1(y)}{1!}\right. \\
&\qquad \left. + \frac{(-r)^{q-2}}{(q-2)!}\cdot\frac{T_{p-q+2}(x)}{(p-q+2)!}\cdot\frac{T_2(y)}{2!} + \ldots\right\}z \quad \text{.....(66).}
\end{aligned}$$

It is more convenient to have the series in the reversed order. The $s+1$th term is

$$\frac{T_{p-q+s}(x)}{(p-q+s)!}\cdot\frac{T_s(z)}{s!}\cdot\frac{(-r)^{q-s}}{(q-s)!}z,$$

and the last term, i.e. when $s = q$, is

$$\frac{T_p(x)}{p!} \cdot \frac{T_q(y)}{q!} z.$$

Hence we may write

$$\frac{\phi^p \psi^q}{p!\,q!}(z) = z\left\{\frac{T_p(x)}{p!} \cdot \frac{T_q(y)}{q!} - \frac{T_{p-1}(x) \cdot T_{q-1}(y)}{(p-1)!\,(q-1)!}\frac{r}{1!} + \frac{T_{p-2}(x) \cdot T_{q-2}(y)}{(p-2)!\,(q-2)!}\frac{r^2}{2!} - \ldots\right\} \quad (67).$$

If now we write $\qquad \phi^p \psi^q(z) = T_{p,q}(x, y) \cdot z,$

so that $T_{p,q}(x, y)$ might be termed a polychoric function in two variables or better a double polychoric function, we shall have

$$\frac{T_{p,q}(x, y)}{p!\,q!} = \frac{T_p(x)}{p!} \cdot \frac{T_q(y)}{q!} - \frac{T_{p-1}(x)}{(p-1)!} \cdot \frac{T_{q-1}(y)}{(q-1)!}\frac{r}{1!} + \frac{T_{p-2}(x)}{(p-2)!} \cdot \frac{T_{q-2}(y)}{(q-2)!} \cdot \frac{r^2}{2!} - \ldots$$

$$= \sum_{s=0}^{p \text{ or } q} \frac{T_{p-s}(x)}{(p-s)!} \cdot \frac{T_{q-s}(y)}{(q-s)!} \frac{(-r)^s}{s!} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (68),$$

p or q being taken as the upper limit of s, whichever is the greater. When $q = 0$

$$\frac{T_{p,0}(x, y)}{p!\,0!} = \frac{T_p(x)}{p!} \cdot \frac{T_0(y)}{0!} = \frac{T_p(x)}{p!} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (69),$$

so that the single polychoric function is the particular case of a double polychoric function, viz. when the order with respect to one of the variables is zero.

The above expansion may be reversed in a manner similar to the process in equations (54) to (58) giving the result

$$\frac{T_p(x)}{p!}\frac{T_q(y)}{q!} = \frac{T_{p,q}(x, y)}{p!\,q!} + \frac{T_{p-1,q-1}(x, y)}{(p-1)!\,(q-1)!}\frac{r}{1!} + \frac{T_{p-2,q-2}(x, y)}{(p-2)!\,(q-2)!} \cdot \frac{r^2}{2!} + \ldots \quad (70).$$

7. *Some properties of the Polychoric Functions.*

We have already seen (53) that

$$\frac{T_n}{n!} + p\frac{T_{n-2}}{2\,(n-2)!} + p^2\frac{T_{n-4}}{2^2.2!\,(n-4)!} + \ldots$$

$$= \frac{x^n}{n!} - (1-p)\frac{x^{n-2}}{2\,(n-2)!} + (1-p)^2\frac{x^{n-4}}{2^2.2!\,(n-4)!} - \ldots \quad \ldots\ldots (71).$$

Putting $\sqrt{1-p} = q$ and therefore $p = 1 - q^2$ we have

$$\frac{T_n}{n!} + (1-q^2)\frac{T_{n-2}}{2\,(n-2)!} + (1-q^2)^2\frac{T_{n-4}}{2^2.2!\,(n-4)!} + \ldots$$

$$= \frac{x^n}{n!} - \frac{x^{n-2}q^2}{2\,(n-2)!} + \frac{x^{n-4}q^4}{2^2.2!\,(n-4)!} - \ldots$$

$$= q^n\left\{\frac{\left(\frac{x}{q}\right)^n}{n!} - \frac{\left(\frac{x}{q}\right)^{n-2}}{2\,(n-2)!} + \frac{\left(\frac{x}{q}\right)^{n-4}}{2^2.2!\,(n-4)!}\ldots\right\}$$

$$= q^n\frac{T_n\left(\frac{x}{q}\right)}{n!} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (72).$$

Similarly by putting $\frac{1}{\sqrt{1-p}} = q$ we have $p = -\frac{1-q^2}{q^2}$ and

$$\frac{T_n}{n!} - \frac{1-q^2}{q^2}\frac{T_{n-2}}{2(n-2)!} + \left(\frac{1-q^2}{q^2}\right)^2 \frac{T_{n-4}}{2^2.2!(n-4)!} - \ldots = \frac{1}{q^n}\frac{T_n(qx)}{n!} \quad \ldots\ldots(73).$$

The following developments may be easily proved by expansion and simple addition:

$$\frac{T_n(x)}{n!} - \frac{x^2 T_{n-2}(x)}{2(n-2)!} + \frac{x^4 T_{n-4}(x)}{2^2.2!(n-4)!} - \ldots = \frac{x^n T_n(1)}{n!} - \frac{x^{n-2} T_{n-2}(1)}{2(n-2)!} + \frac{x^{n-4} T_{n-4}(1)}{2^2.2!(n-4)!} \quad \ldots\ldots(74),$$

$$\frac{T_n(x+a)}{n!} = \frac{x^n}{n!} + \frac{x^{n-1} T_1(a)}{(n-1)!} + \frac{x^{n-2} T_2(a)}{2!(n-2)!} + \ldots \quad \ldots\ldots\ldots\ldots\ldots(75).$$

Since we have the binomial coefficients multiplied into terms of the type $x^{n-s} T_s(a)$ we may write this symbolically

$$T_n(x+a) = \{x + T(a)\}^n,$$

with the convention that the x's will have indices and the T's suffixes. Clearly we may also write

$$T_n(x+a) = (T(x) + a)^n.$$

Similarly

$$\frac{(x+a)^n}{n!} = \frac{T_n(x)}{n!} + \frac{T_{n-1}(x).\mathfrak{A}_1(a)}{(n-1)!1!} + \frac{T_{n-2}(x).\mathfrak{A}_2(a)}{(n-2)!2!} + \ldots \quad \ldots\ldots(76).$$

Therefore

$$\begin{aligned}(x+a)^n &= \{T(x) + \mathfrak{A}(a)\}^n \\ &= \{\mathfrak{A}(x) + T(a)\}^n,\end{aligned}$$

with the same convention as before as to indices and suffixes.

The following is of interest. Write $\theta_n = \frac{T_n(x)}{n!}$, then

$$\begin{aligned}\frac{d\theta_n}{dx} &= \frac{d}{dx}\left(\frac{x^n}{n!} - \frac{x^{n-2}}{2.(n-2)!} + \frac{x^{n-4}}{2^2.2!(n-4)!} - \ldots\right) \\ &= \frac{x^{n-1}}{(n-1)!} - \frac{x^{n-3}}{2.(n-3)!} + \frac{x^{n-5}}{2^2.2!(n-5)!} - \ldots \\ &= \frac{T_{n-1}(x)}{(n-1)!} = \theta_{n-1} \quad \ldots\ldots(77).\end{aligned}$$

Since

$$\begin{aligned}T_n &= xT_{n-1} - (n-1)T_{n-2}, \\ \theta_n &= \frac{1}{n}(x\theta_{n-1} - \theta_{n-2}) \\ &= \frac{1}{n}\left(x\frac{d\theta_n}{dx} - \frac{d^2\theta_n}{dx^2}\right) \\ &= \frac{1}{n}\left(x - \frac{d}{dx}\right)\frac{d}{dx}\theta_n \\ &= \frac{1}{n}\left(x - \frac{d}{dx}\right)\theta_{n-1} \\ &= \frac{1}{n}\left(x - \frac{d}{dx}\right)\left(x - \frac{d}{dx}\right)\frac{\theta_{n-2}}{n-1} \\ &= \ldots\ldots\ldots\ldots\ldots\ldots \\ &= \frac{1}{n!}\left(x - \frac{d}{dx}\right)^n .1 \quad \ldots\ldots(78).\end{aligned}$$

Various functions may be expanded in terms of T. Thus since

$$T_n . z = \phi^n(z) = \left(-\frac{d}{dx}\right)^n (e^{-\frac{1}{2}x^2}) = (-)^n \frac{d^n}{dx^n} e^{-\frac{1}{2}x^2},$$

$$e^{-\frac{1}{2}(x-1)^2} = f(x-1) = f_x - \frac{df}{dx} + \frac{1}{2!}\cdot\frac{d^2f}{dx^2} - \ldots$$

$$= e^{-\frac{1}{2}x^2}\left(\frac{T_0}{0!} + \frac{T_1}{1!} + \frac{T_2}{2!} + \ldots\right)$$

$$= e^{-\frac{1}{2}x^2}(\theta_0 + \theta_1 + \theta_2 + \ldots) \quad \ldots\ldots\ldots\ldots(79).$$

$$\therefore\ \theta_0 + \theta_1 + \theta_2 + \ldots = e^{-\frac{1}{2}(x-1)^2 + \frac{1}{2}x^2} = e^{x - \frac{1}{2}} \quad \ldots\ldots\ldots\ldots(80).$$

$$\therefore\ e^x = e^{\frac{1}{2}}(\theta_0 + \theta_1 + \theta_2 + \ldots) \quad \ldots\ldots\ldots\ldots(81).$$

8. *Expansion of the Polychoric Function.*

If we write T_p, T_q' for $T_p(x)$ and $T_q(y)$ respectively we have seen that (1) T_p may be written T_{p0}, (2) T_q' as T_{0q}. The same symbol T may be used for the single or the double function, as its order is shown by the number of suffixes.

$$\frac{T_{pq}}{p!\,q!} = \frac{T_p T_q'}{p!\,q!} - r\frac{T_{p-1}T'_{q-1}}{(p-1)!(q-1)!} + \frac{r^2}{2!}\cdot\frac{T_{p-2}T'_{q-2}}{(p-2)!\,(q-2)!} - \frac{r^3}{3!}\cdot\frac{T_{p-3}T'_{q-3}}{(p-3)!(q-3)!}$$

$$= \frac{T_p T_q'}{p!\,q!} - \int\left\{\frac{T_{p-1}T'_{q-1}}{(p-1)!(q-1)!} - r\frac{T_{p-2}T'_{q-2}}{(p-2)!(q-2)!} + \frac{r^2}{2!}\cdot\frac{T_{p-3}T'_{q-3}}{(p-3)!(q-3)!} - \ldots\right\} dr$$

$$= \frac{T_p T_q'}{p!\,q!} - \int\frac{T_{p-1,q-1}}{(p-1)!(q-1)!}\,dr \quad \ldots\ldots\ldots\ldots(82),$$

and

$$T_{pq} = T_p T_q' - p\,.\,q\int T_{p-1,q-1}\,dr \quad \ldots\ldots\ldots\ldots(83),$$

which gives a simple method of expanding T_{pq}.

It may be convenient on occasion to express the above in the θ notation

$$\theta_{pq} = \theta_p\theta_q' - \int\theta_{p-1,q-1}\,dr \quad \ldots\ldots\ldots\ldots(84).$$

Having expanded T_{pq} we can at once write down the expansion $\phi^p\psi^q$. The two following tables give the expansions required, viz. $\phi^p\psi^q$ in terms of powers of x and y multiplied by z and its converse.

TABLE I.

xy products of z in terms of polychoric functions.

1. $xz = \phi(z)$
2. $x^2z = (\phi^2 + 1)z$
   $xyz = (\phi\psi + r)z$
3. $x^3z = (\phi^3 + 3\phi)z$
   $x^2yz = (\phi^2\psi + \psi + 2r\phi)z$

TABLE I (*continued*).

4. $x^4z = (\phi^4 + 6\phi^2 + 3)\,z$

$x^3yz = (\phi^3\psi + 3\phi\psi + 3r\phi^2 + 3r)\,z$

$x^2y^2z = (\phi^2\psi^2 + \phi^2 + 4r\phi\psi + \psi^2 + 2r^2 + 1)\,z$

5. $x^5z = (\phi^5 + 10\phi^3 + 15\phi)\,z$

$x^4yz = (\phi^4\psi + 6\phi^2\psi + 3\psi + 12r\phi + 4r\phi^3)\,z$

$x^3y^3z = (\phi^3\psi^3 + 3\phi\psi^2 + 6r\phi^2\psi + 6r\psi + 3(1 + 2r^2)\,\phi + \phi^3)\,z$

6. $x^6z = (\phi^6 + 15\phi^4 + 45\phi^2 + 15)\,z$

$x^5yz = (\phi^5\psi + 10\phi^3\psi + 15\phi\psi + 30r\phi^2 + 5r\phi^4 + 15r)\,z$

$x^4y^2z = (\phi^4\psi^2 + 6\phi^2\psi^2 + 3\psi^2 + 24r\phi\psi + 8r\phi^3\psi + 6(1 + 2r^2)\,\phi^2 + \phi^4 + 3(1 + 4r^2))\,z$

$x^3y^3z = (\phi^3\psi^3 + 3\phi^3\psi + 9r\phi^2\psi^2 + 9r\phi^3 + 3\phi\psi^2 + 9(1 + 2r^2)\,\phi\psi + 9r\psi^2 + 3r(3 + 2r^2))\,z$

7. $x^7z = (\phi^7 + 21\phi^5 + 105\phi^3 + 105\phi)\,z$

$x^6yz = (\phi^6\psi + 15\phi^4\psi + 45\phi^2\psi + 15\psi + 90r\phi + 60r\phi^3 + 6r\phi^5)\,z$

$x^5y^2z = (\phi^5\psi^2 + 10\phi^3\psi^2 + 15\phi\psi^2 + 10r\phi^4\psi + 60r\phi^2\psi + 30r\psi + 15(1 + 4r^2)\,\phi + 10(1 + 2r^2)\,\phi^3 + \phi^5)\,z$

$x^4y^3z = (\phi^4\psi^3 + 6\phi^2\psi^3 + 3\psi^3 + 12r\phi^3\psi^2 + 36r\phi\psi^2 + 3\phi^4\psi + 18(1 + 2r^2)\,\phi^2\psi + 9(1 + 4r^2)\,\psi + 12r(3 + 2r^2)\,\phi + 12r\phi^3)\,z$

8. $x^8z = (\phi^8 + 28\phi^6 + 210\phi^4 + 420\phi^2 + 105)\,z$

$x^7yz = (\phi^7\psi + 21\phi^5\psi + 105\phi^3\psi + 105\phi\psi + 315r\phi^2 + 105r\phi^4 + 7r\phi^6 + 105r)\,z$

$x^6y^2z = (\phi^6\psi^2 + 15\phi^4\psi^2 + 45\phi^2\psi^2 + 15\psi^2 + 180r\phi\psi + 120r\phi^3\psi + 12r\phi^5\psi + 45(1 + 4r^2)\,\phi^2 + 15(1 + 2r^2)\,\phi^4 + \phi^6 + 15(1 + 6r^2))\,z$

$x^5y^3z = (\phi^5\psi^3 + 3\phi^5\psi + 30(1 + 2r^2)\,\phi^3\psi + 45(1 + 4r^2)\,\phi^4 + 30r(3 + 2r^2)\,\phi^2 + 15r\phi^4 + 10\phi^2\psi^3 + 90r\phi^2\psi^2 + 15\phi\psi^3 + 45r\psi^2 + 15r\phi^4\psi^2 + 15r(3 + 4r^2))\,z$

$x^4y^4z = (\phi^4\psi^4 + 6\phi^4\psi^2 + 6\phi^2\psi^4 + 36(1 + 2r^2)\,\phi^2\psi^2 + 3\phi^4 + 3\psi^4 + 48r\phi^3\psi + 48r\phi\psi^3 + 18(1 + 4r^2)\,\phi^2 + 18(1 + 4r^2)\,\psi^2 + 48r(3 + 2r^2)\,\phi\psi + 16r\phi^3\psi^3 + 3(3 + 24r^2 + 8r^4))\,z.$

TABLE II.

Polychoric functions expanded in terms of x and y.

1. $\phi(z) = xz$

2. $\phi^2(z) = (x^2 - 1)\,z$

$\phi\psi(z) = (xy - r)\,z$

3. $\phi^3(z) = (x^3 - 3x)\,z$

$\phi^2\psi(z) = (x^2y - y - 2rx)\,z$

4. $\phi^4(z) = (x^4 - 6x^2 + 3)\,z$

$\phi^3\psi(z) = (x^3y - 3rx^2 - 3xy + 3r)\,z$

$\phi^2\psi^2(z) = (x^2y^2 - x^2 - y^2 - 4rxy + 1 + 2r^2)\,z$

5. $\phi^5(z) = (x^5 - 10x^3 + 15x)\,z$

$\phi^4\psi(z) = (x^4y - 6x^2y + 3y - 4rx^3 + 12rx)\,z$

$\phi^3\psi^2(z) = (x^3y^2 - 6rx^2y - 3xy^2 - x^3 + 3(1 + 2r^2)\,x + 6ry)\,z$

TABLE II (*continued*).

6. $\phi^0(z) = (x^6 - 15x^4 + 45x^2 - 15)\,z$

$\phi^5\psi(z) = (x^5y - 10x^3y + 15xy - 15r - 5rx^4 + 30rx^2)\,z$

$\phi^4\psi^2(z) = (x^4y^2 - 6x^2y^2 - 8rx^3y + 6(1 + 2r^2)\,x^2 + 3y^2 + 24rxy - x^4 - 3(1 + 4r^2))\,z$

$\phi^3\psi^3(z) = (x^3y^3 - 3x^3y - 3xy^3 - 9rx^2y^2 + 9(1 + 2r^2)\,xy + 9rx^2 + 9ry^2 - 3r(3 + 2r^2))\,z$

7. $\phi^7(z) = (x^7 - 21x^5 + 105x^3 - 105x)\,z$

$\phi^6\psi(z) = (x^6y - 15x^4y + 45x^2y - 15y - 6x^5r + 60x^3r - 90xr)\,z$

$\phi^5\psi^2(z) = (x^5y^2 - 10x^3y^2 + 15xy^2 - 10rx^4y + 60rx^2y - 30ry$
$\quad - 15(1 + 4r^2)\,x + 10(1 + 2r^2)\,x^3 - x^5)\,z$

$\phi^4\psi^3(z) = (x^4y^3 - 6x^2y^3 + 3y^3 - 12rx^3y^2 + 36rxy^2 - 3x^4y$
$\quad + 18(1 + 2r^2)\,x^2y - 9(1 + 4r^2)\,y - 12r(3 + 2r^2)\,x + 12rx^3)\,z$

8. $\phi^8(z) = (x^8 - 28x^6 + 210x^4 - 420x^2 + 105)\,z$

$\phi^7\psi(z) = (x^7y - 21x^5y + 105x^3y - 105xy - 315rx^3 + 105rx^4 - 7rx^6 + 105r)\,z$

$\phi^6\psi^2(z) = (x^6y^2 - 15x^4y^2 + 45x^2y^2 - 15y^2 - 180rxy + 120rx^2y$
$\quad - 12rx^5y - 45(1 + 4r^2)\,x^2 + 15(1 + 2r^2)\,x^4 - x^6 + 15(1 + 6r^2))\,z$

$\phi^5\psi^3(z) = (x^5y^3 - 3x^5y + 30(1 + 2r^2)\,x^3y - 45(1 + 4r^2)\,xy - 30r(3 + 2r^2)\,x^3$
$\quad + 15rx^4 - 10x^3y^3 + 90rx^2y^2 + 15xy^3 - 45ry^3 - 15rx^4y^2 + 15r(3 + 4r^2))\,z$

$\phi^4\psi^4(z) = (x^4y^4 - 6x^4y^2 - 6x^2y^4 + 36(1 + 2r^2)\,x^2y^2 + 3x^4 + 3y^4$
$\quad + 48rx^3y + 48rxy^3 - 18(1 + 4r^2)\,x^2 - 18(1 + 4r^2)\,y^2$
$\quad - 48r(3 + 2r^2)\,xy - 16rx^2y^3 + 3(3 + 24r^2 + 8r^4))\,z.$

The above formulae may be immediately expressed in terms of T_{pq} by remembering that $\phi^p\psi^q(z) = T_{pq}\,z$.

Thus from $x^2yz = (\phi^2\psi + \psi + 2r\phi)\,z$

we get $x^2y = T_{21} + T_{01} + 2rT_{10}$

and from $\phi^2\psi(z) = (x^2y - y - 2rx)\,z$

we get $T_{21} = x^2y - y - 2rx.$

9. *Moments of a quadrant.*

From the preceding tables we may express any moments of z in sums of functions involving ϕ and ψ. Consider one term of such a sum:

$$\int_{-\infty}^{k}\int_{-\infty}^{h}\phi^p\psi^q\,z\,(x,y)\,dx\,dy$$

$$= -\int_{-\infty}^{k}\int_{-\infty}^{h}\phi^{p-1}\psi^q\left(\frac{d}{dx} + r\frac{d}{dy}\right)z\,(x,y)\,dx\,dy$$

$$= -\int_{-\infty}^{k}\int_{-\infty}^{h}\phi^{p-1}\psi^q\frac{d}{dx}\,z\,(x,y)\,dx\,dy - r\int_{-\infty}^{k}\int_{-\infty}^{h}\phi^{p-1}\psi^q\frac{d}{dy}\,z\,(x,y)\,dx\,dy$$

$$= -\int_{-\infty}^{k}\phi^{p-1}\psi^q\,z\,(h,y)\,dy - r\int_{-\infty}^{h}\phi^{p-1}\psi^q\,z\,(x,k)\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(85).$$

These when expanded by means of the tables will give a series of terms such as

$$(a)\ \int_{-\infty}^{k} h^l y^m z(h, y)\, dy \quad \text{and} \quad (b)\ \int_{-\infty}^{h} x^{l'} k^{m'} z(x, k)\, dx.$$

(a) may be regarded as the (lm)th moment of $\int_{-\infty}^{k} z(h, y)\, dy$, that is of HA (see equation 7), and (b) may be regarded as the $(l'm')$th moment of $\int_{-\infty}^{h} z(x, k)dx$, that is of KB. That is they are moments of the bounding surfaces of the quadrant (equation 8).

If we write the moment coefficients of HA and KB as $\bar{A}_{lm}$ and $\bar{B}_{l'm'}$ respectively we may write

$$\int_{-\infty}^{k} h^l y^m z(h, y)\, dy = HA\,.\,\bar{A}_{lm} = HA\,.\,h^l \bar{A}_{0m} \quad \text{.....(86),}$$

$$\int_{-\infty}^{h} x^{l'} k^{m'} z(x, k)\, dx = KB\,.\,\bar{B}_{l'm'} = KB\,.\,k^{m'} \bar{B}_{l'0} \quad \text{.....(87).}$$

It remains therefore to find a means of evaluating $\bar{A}_{0m}$ and $\bar{B}_{l'0}$.

An alternative form of the expansion will be got by expanding a ψ instead of a ϕ but the ultimate results will be identical.

To take an actual example

$$\begin{aligned}
&\int_{-\infty}^{k}\int_{-\infty}^{h} x^2 yz\, dx\, dy \\
&= \int_{-\infty}^{k}\int_{-\infty}^{h} (\phi^2\psi + \psi + 2r\phi)\, z\, dx\, dy \\
&= -\int_{-\infty}^{k}\int_{-\infty}^{h} \left\{(\phi^2 + 1)\left(r\frac{d}{dx} + \frac{d}{dy}\right) + 2r\left(\frac{d}{dx} + r\frac{d}{dy}\right)\right\} z\, dx\, dy \\
&= -\int_{-\infty}^{k}\int_{-\infty}^{h} \left\{(r\phi^2 + r + 2r)\frac{d}{dx} + (\phi^2 + 1 + 2r^2)\frac{d}{dy}\right\} z\, dx\, dy \\
&= -\int_{-\infty}^{k} (r\phi^2 + 3r)\, z(h, y)\, dy - \int_{-\infty}^{h} (\phi^2 + 1 + 2r^2)\, z(x, k)\, dx \\
&= -\int_{-\infty}^{k} (rh^2 - r + 3r)\, z(h, y)\, dy - \int_{-\infty}^{h} (x^2 - 1 + 1 + 2r^2)\, z(x, k)\, dx \\
&= -r\int_{-\infty}^{k} h^2 z(h, y)\, dy - 2r\int_{-\infty}^{k} z(h, y)\, dy - \int_{-\infty}^{h} x^2 z(x, k)\, dx - 2r^2\int_{-\infty}^{h} z(x, k)\, dx \\
&= -rh^2 HA\,.\, - 2rH\,.\,A - KB\,.\,\bar{B}_{20} - 2r^2 KB \\
&= -HA\,(rh^2 + 2r) - KB\,(\bar{B}_{20} + 2r^2) \quad \text{.....(88).}
\end{aligned}$$

For theoretical purposes the development may be treated more systematically as follows.

Since $$\int_{-\infty}^{k} h^l y^m z(h, y)\, dy = HA\,.\,\bar{A}_{lm} \quad \text{.....(89),}$$

it is clear that $$\int_{-\infty}^{k} \phi^{p-1} q\, z(h, y)\, dx = \int_{-\infty}^{k} T_{p-1,q}(h, y)\, z(h, y)\, dx$$
$$= HA\,.\,T_{p-1,q}\bar{A} \quad \text{.....(90),}$$

in which the T will be expanded in A's having the same suffixes as the h's and y's.

Hence we may write immediately

$$\int_{-\infty}^{h}\int_{-\infty}^{h}\phi^p\psi^q z\,dx\,dy = -HA\,.\,T_{p-1,q}\overline{A} - rKB\,.\,T_{p-1,q}\overline{B} = -T_{p-1,q}M \quad \ldots(91),$$

or reducing the ψ

$$= -rHA\,.\,T_{p,q-1}\overline{A} - KB\,.\,T_{p,q-1}\overline{B} = -T_{p,q-1}M',$$

where M or M' may be regarded as a kind of complex moment coefficient.

Applying this to our example we have

$$\begin{aligned}&\int_{-\infty}^{h}\int_{-\infty}^{h}(\phi^2\psi + \psi + 2r\phi)\,z\,dx\,dy\\ &= -\{T_{20}(M') + T_0(M') + 2rT_0M\}\\ &= -\{rHA\,.\,T_{20}(\overline{A}) + KB\,.\,T_{20}\overline{B}\}\\ &\quad -(rHA + KB) - 2r(HA + rKB) \quad \ldots\ldots\ldots\ldots\ldots(92).\end{aligned}$$

But $$T_{20}(\overline{A}) = \overline{A}_{20} - 1 = h^2 - 1$$

and $$T_{20}(\overline{B}) = \overline{B}_{20} - 1,$$

and the whole reduces as before to

$$-HA(rh^2 + 2r) - KB(\overline{B}_{20} + 2r^2).$$

10. *Evaluation of the moments of the bounding surfaces of the quadrant.*

In equation (7) the area of the surface bounding the quadrant a and parallel to the y axis was represented by the symbol

$$N\frac{H}{\sigma_x}A = N\int_{-\infty}^{k'} z'(k', y')\,dy',$$

and that parallel to the x axis by

$$N\frac{K}{\sigma_y}B = N\int_{-\infty}^{h'} z'(x', k')\,dx'.$$

Consider now the pqth moment of say the latter, the B face. It will be represented by

$$N\int_{-\infty}^{h'} x'^p k'^q z'(x', k')\,dx' = Nk'^q\int_{-\infty}^{h'} x'^p z'(x', k')\,dx' \quad \ldots\ldots\ldots\ldots(93),$$

since $y' = k'$ over the whole face. Representing this as the product of the area and its pqth moment coefficient we may write

$$N\int_{-\infty}^{h'} x'^p k'^q z'(x', k')\,dx' = N\frac{K}{\sigma_y}B\,.\,\overline{B}'_{pq} = N\frac{K}{\sigma_y}B\,.\,k'^q\,.\,\overline{B}'_{p,0} \quad \ldots\ldots\ldots(94),$$

and it is the value of the coefficient $\overline{B}'_{p,0}$ that we have to determine

$$\begin{aligned}\frac{d}{dx'}\{x'^n\,.\,z'(x', k')\} &= nx'^{n-1}\,.\,z'(x', k') + x'^n\left\{-\frac{x'/\sigma_x - ry'/\sigma_y}{\sigma_x(1-r^2)}\right\}z'(x', k')\\ &= \left(nx'^{n-1} - \frac{x'^{n+1}}{\sigma_x^2(1-r^2)} - r\frac{k'}{\sigma_y}\cdot\frac{x'^n}{\sigma_x(1-r^2)}\right)z'(x', k') \quad \ldots(95).\end{aligned}$$

Transposing and integrating

$$\frac{1}{\sigma_x^2(1-r^2)}\int_{-\infty}^{h'} x'^{n+1}.z'(x',k')\,dx' = n\int_{-\infty}^{h'} x'^{n-1}.z'(x',k')\,dx' + \frac{rk'}{\sigma_x\sigma_y(1-r^2)}\int_{-\infty}^{h'} x'^{n}.z'(x',k')\,dx' - \Big|_{-\infty}^{h'} x'^{n}.z'(x',k') \quad \ldots\ldots(96),$$

that is,

$$\frac{1}{\sigma_x^2(1-r^2)}\cdot\frac{KB}{\sigma_y}\cdot\bar{B}'_{n+1,0} = n\frac{KB}{\sigma_y}\bar{B}'_{n-1,0} + \frac{rk'}{\sigma_x\sigma_y(1-r^2)}\cdot\frac{KB}{\sigma_y}\bar{B}'_{n,0} - \frac{h'^n\chi}{\sigma_x\sigma_y} \ldots(97),$$

where $\chi = z(h, k)$, see equation (10).

Reducing by one degree and dividing out by $\dfrac{KB}{\sigma_y\sigma_x^2(1-r^2)}$,

$$\bar{B}'_{n,0} = \frac{rk'\sigma_x}{\sigma_y}\bar{B}'_{n-1,0} + (n-1)(1-r^2)\sigma_x^2\bar{B}'_{n-2,0} - \frac{h'^{n-1}\sigma_x\chi}{KB}(1-r^2) \ldots(98).$$

Hence

$$\bar{B}'_{10} = \frac{rk'\sigma_x}{\sigma_y} - \frac{\sigma_x\chi}{KB}(1-r^2) \quad \ldots\ldots(99),$$

and we may write

$$\bar{B}'_{n,0} = \frac{rk'\sigma_x}{\sigma_y}\bar{B}'_{n-1,0} + (n-1)(1-r^2)\sigma_x^2\bar{B}'_{n-2,0} + h'^{n-1}\left(\bar{B}'_{10} - \frac{rk'\sigma_x}{\sigma_y}\right) \ldots(100).$$

Divide throughout by $\sigma_x^n(\sqrt{1-r^2})^n$ and write

$$\sigma_x\sqrt{1-r^2} = S_x,$$
$$\sigma_y\sqrt{1-r^2} = S_y,$$

$$\frac{\bar{B}'_{n,0}}{S_x^n} = \frac{rk'}{S_y}\cdot\frac{\bar{B}'_{n-1,0}}{S_x^{n-1}} + (n-1)\frac{\bar{B}'_{n-2,0}}{S_x^{n-2}} + \frac{h'^{n-1}}{S_x^{n-1}}\cdot\frac{\bar{B}'_{10}}{S_x} - \frac{rk'}{S_y}\cdot\frac{h'^{n-1}}{S_x^{n-1}} \quad \ldots\ldots(101).$$

Let now

$$\frac{\bar{B}'_{n,0}}{S_x^n} = \beta_n; \quad \frac{h'}{S_x} = \iota; \quad \frac{k'}{S_y} = \kappa \quad \ldots\ldots(102)$$

then

$$\beta_n = r\kappa\beta_{n-1} + (n-1)\beta_{n-2} + \iota^{n-1}\beta_1 - r\iota^{n-1}\kappa \quad \ldots\ldots(103)$$

Putting $n = 1$ in (81),

$$\beta_1 = \frac{\bar{B}_{10}}{\sigma_x\sqrt{1-r^2}} = \frac{rk'}{\sigma_y\sqrt{1-r^2}} - \frac{\chi}{KB}(\sqrt{1-r^2})$$

$$= \frac{rk'}{\sigma_y\sqrt{1-r^2}} - \frac{1}{\sqrt{1-r^2}}\cdot\frac{E(k).E\left(\dfrac{h-rk}{\sqrt{1-r^2}}\right)}{K\mathscr{E}\left(\dfrac{h-rk}{\sqrt{1-r^2}}\right)}\sqrt{1-r^2}$$

$$= \frac{rk'}{\sigma_y\sqrt{1-r^2}} - \frac{E\left(\dfrac{h'/\sigma_x - rk'/\sigma_y}{\sqrt{1-r^2}}\right)}{\mathscr{E}\left(\dfrac{h'/\sigma_x - rk'/\sigma_y}{\sqrt{1-r^2}}\right)}$$

$$= \frac{rk'}{\sigma_y\sqrt{1-r^2}} - \frac{E(\iota - r\kappa)}{\mathscr{E}(\iota - r\kappa)}$$

$$= r\kappa + c_\beta \quad \ldots\ldots(104),$$

where c_β is the centroid of the face B. See equation (16).

Substituting in (85),

$$\beta_n = r\kappa\beta_{n-1} + (n-1)\,\beta_{n-2} + \iota^{n-1}c_\beta \quad \text{.....(105).}$$

Similarly for the A face

$$\int_{-\infty}^{h'} h'^p y'^q z'(h', y')\,dy' = N\frac{H}{\sigma_x} A \,.\, h'^p \,.\, \overline{A}_{0q},$$

$$\frac{\overline{A}'_{0n}}{S_y^{\,n}} = r\frac{h'}{S_x}\cdot\frac{\overline{A}'_{0,n-1}}{S_y^{\,n-1}} + (n-1)\frac{\overline{A}'_{0,n-2}}{S_y^{\,n-2}} + \frac{k'^{n-1}}{S_y^{\,n-1}}\cdot\frac{\overline{A}'_{01}}{S_y} - r\frac{h'}{S_x}\cdot\frac{k'^{n-1}}{S_y^{\,n-1}} \quad \text{... (106),}$$

$$\frac{\overline{A}'_{0n}}{S_y^{\,n}} = a_n,$$

$$a_1 = \frac{rh'}{\sigma_x\sqrt{1-r^2}} - \frac{E(\kappa - r\iota)}{\mathscr{E}(\kappa - r\iota)} = r\iota + c_a \quad \text{.....(107),}$$

$$a_n = r\iota a_{n-1} + (n-1)\,a_{n-2} + \kappa^{n-1}c_a \quad \text{.....(108).}$$

11. *The moments in terms of the tetrachoric coefficients.*

We have already seen (77) that

$$\frac{d\theta_p(x)}{dx} = \theta_{p-1}(x) \quad \text{and} \quad \frac{d\theta_q(y)}{dy} = \theta_{q-1}(y).$$

Drop the arguments and write θ_p, θ_q' for $\theta_p(x)$, $\theta_q(y)$ respectively,

$$\frac{d\theta_{pq}}{dx} = \frac{d}{dx}\left(\theta_p\theta_q' - r\theta_{p-1}\theta'_{q-1} + \frac{r^2}{2!}\cdot\theta_{p-2}\theta'_{q-2}\cdots\right)$$

$$= \theta_{p-1}\theta_q' - r\theta_{p-2}\theta'_{q-1} + \frac{r^2}{2!}\cdot\theta_{p-3}\theta'_{q-2}\cdots$$

$$= \theta_{p-1,q} \quad \text{.....(109).}$$

Similarly

$$\frac{d\theta_{p-1,q}}{dy} = \theta_{p-1,q-1},$$

$$\therefore \quad \frac{d\theta_{p,q}}{dx\,dy} = \theta_{p-1,q-1} \quad \text{.....(110).}$$

But

$$\frac{d\theta_{p,q}}{dr} = \frac{d}{dr}\left(\theta_p\theta_q' - r\theta_{p-1}\theta'_{q-1} + \frac{r^2}{2!}\theta_{p-2,q-2}\cdots\right)$$

$$= -\theta_{p-1}\theta'_{q-1} + r\theta_{p-2,q-2} - \frac{r^2}{2!}\theta_{p-3,q-3}$$

$$= -\theta_{p-1,q-1}$$

$$= -\frac{d\theta_{p,q}}{dx\,dy} \quad \text{.....(111).}$$

Hence

$$\left(\frac{d\theta_{pq}}{dr}\right)_{r=0} = -\theta_{p-1}\theta'_{q-1} \quad \text{.....(112).}$$

The result in (111) corresponds to that deduced by Pearson (in *Phil. Trans.* Vol. 195, pp. 1–47), viz.

$$\frac{dz}{dr} = \frac{d^2z}{dx\,dy} \quad \text{.....(113).}$$

We have further, since

$$\frac{dz}{dx} = -\frac{(\phi - r\psi)}{1-r^2}(z), \quad \frac{dz}{dy} = -\frac{(\psi - r\phi)}{1-r^2}(z) \quad \text{.....(114),}$$

$$\frac{d^n z}{dr^n} = \frac{d^{2n} z}{dx^n dy^n} = \frac{(\phi - r\psi)^n (\psi - r\phi)^n}{(1-r^2)^{2n}}(z) \text{.....(115).}$$

Hence

$$\left(\frac{d^n z}{dr^n}\right)_{r=0} = (\phi^n \psi^n z)_{r=0}$$

$$= (T_{nn'} z)_{r=0}$$

$$= T_n T_n' z_0 \quad \text{.....(116),}$$

where $z_0 = (z)_{r=0}$.

Hence, developing z as a function of r,

$$z = z_0 + r\left(\frac{dz}{dr}\right)_{r=0} + \frac{r^2}{2!}\left(\frac{d^2 z}{dr^2}\right)_{r=0} + \dots$$

$$= z_0 \left(1 + r T_1 T_1' + \frac{r^2}{2!} T_2 T_2' + \dots\right) \text{.....(117).}$$

Therefore

$$\frac{\phi^p \psi^q z}{p!\,q!} = \theta_{pq} z = \theta_{pq}\left(1 + r T_1 T_1' + \frac{r^2}{2!} T_2 T_2' + \dots\right) z_0$$

$$= \left(\theta_p \theta_q' - r\theta_{p-1}\theta'_{q-1} + \frac{r^2}{2!}\theta_{p-2}\theta'_{q-2} \dots\right)\left(1 + r T_1 T_1' + \frac{r^2}{2!} T_2 T_2' + \dots\right) z_0$$

$$= \theta_p \theta_q' z_0$$

$$+ r(\theta_p \theta_q' T_1 T_1' - \theta_{p-1}\theta'_{q-1}) z_0$$

$$+ \frac{r^2}{2!}(\theta_p \theta_q' T_2 T_2' - 2\theta_{p-1}\theta'_{q-1} T_1 T_1' + \theta_{p-2}\theta'_{q-2}) z_0$$

$$+ \frac{r^3}{3!}(\theta_p \theta_q' T_3 T_3' - 3\theta_{p-1}\theta'_{q-1} T_2 T_2' + 3\theta_{p-2}\theta'_{q-2} T_1 T_1' - \theta_{p-3}\theta'_{q-3}) z_0$$

$$+ \dots \quad \text{.....(118).}$$

The development of the coefficients of the r's is obvious. If we write the coefficient of $\frac{r^n}{n!}$ as $(\theta_{pq} T)_n$, or where there is no possibility of confusion $(\theta T)_n$, we have

$$\frac{\phi^p \psi^q (z)}{p!\,q!} = \left\{\theta_p \theta_q' + r(\theta T)_1 + \frac{r^2}{2!}(\theta T)_2 + \frac{r^3}{3!}(\theta T)_3 + \dots\right\} z_0 \quad \text{.....(119).}$$

Hence

$$\int_{-\infty}^{k}\int_{-\infty}^{h} \frac{\phi^p \psi^q (z)}{p!\,q!}\,dx\,dy = \int_0^r \frac{\phi^p \psi^q (z)}{p!\,q!}\,dr$$

$$= C + \left(\theta_p \theta_q' r + (\theta T)_1 \frac{r^2}{2!} + (\theta T)_2 \frac{r^3}{3!} + (\theta T)_3 \frac{r^4}{4!} + \dots\right) z_0 (h, k) \text{ ...(120),}$$

where

$$z_0 (h, k) = HK.$$

To determine C,

$$\frac{d}{dx}(\theta_n e^{-\frac{1}{2}x^2}) = \frac{d\theta_n}{dx} \cdot e^{-\frac{1}{2}x^2} - \theta_n \cdot xe^{-\frac{1}{2}x^2}$$
$$= -(x\theta_n - \theta_{n-1}) e^{-\frac{1}{2}x^2}$$
$$= -(n+1)\,\theta_{n+1} e^{-\frac{1}{2}x^2} \quad \text{.....(121).}$$

Therefore
$$\int \theta_{n+1} e^{-\frac{1}{2}x^2} dx = -\frac{\theta_n e^{-\frac{1}{2}x^2}}{n+1} \quad \text{.....(122).}$$

Now put $r=0$ in equation (120),

$$C = \int_{-\infty}^{h}\int_{-\infty}^{h} \frac{\phi^p \psi^q}{p!q!} z\,dx\,dy \qquad (r=0)$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{h} \theta_p e^{-\frac{1}{2}x^2} dx \cdot \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{k} \theta_q' e^{-\frac{1}{2}y^2} dy$$
$$= \frac{1}{\sqrt{2\pi}} \frac{\theta_{p-1} e^{-\frac{1}{2}x^2}}{p} \cdot \frac{1}{\sqrt{2\pi}} \frac{\theta'_{q-1} e^{-\frac{1}{2}y^2}}{q}$$
$$= \frac{\theta_{p-1}\theta'_{q-1} e^{-\frac{1}{2}(x^2+y^2)}}{p \cdot q \cdot 2\pi} = \frac{\theta_{p-1}\theta'_{q-1}}{pq} HK \quad \text{.....(123).}$$

Hence finally

$$\int_{-\infty}^{k}\int_{-\infty}^{h} \phi^p \psi^q z\,dx\,dy = p!q!HK\left\{\frac{\theta_{p-1}\theta'_{q-1}}{p \cdot q} + \theta_p\theta_q' r + (\theta T)_1 \frac{r^2}{2!} + (\theta T)_2 \frac{r^3}{3!} + \dots\right\}$$
.....(124).

When $p=q=0$ we get the volume of the quadrant. In this case $\frac{\theta_{p-1}}{p}$ becomes indeterminate, since putting $n=1$ in $\theta_n = \frac{x\theta_{n-1} - \theta_{n-2}}{n}$ we get

$$\theta_1 = x\theta_0 - \theta_{-1} \quad \text{and} \quad \theta_{-1} = x\theta_0 - \theta_1 = x - x = 0.$$

But from (61) we have

$$T_{-1} e^{-\frac{1}{2}x^2} = -\int_{-\infty}^{x} e^{-\frac{1}{2}x^2} dx \quad \text{.....(125).}$$

Hence we may write in this case

$$C = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{h} T_0 e^{-\frac{1}{2}x^2} dx \cdot \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{k} T_0 e^{-\frac{1}{2}y^2} dy$$
$$= T_{-1} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \cdot T'_{-1} \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}$$
$$= T_{-1} T'_{-1} HK \quad \text{.....(126).}$$

In this case $\theta_p = \theta_q' = 1$ and all the other θ's vanish; and we have

$$\int_{-\infty}^{k}\int_{-\infty}^{h} z\,dx\,dy = HK\left\{T_{-1}T'_{-1} + rT_0T_0' + \frac{r^2}{2!}T_1T_1' + \frac{r^3}{3!}T_2T_2' + \dots\right\}$$
.....(127),

which is the ordinary tetrachoric expansion in terms of T.

Applying the above to our former example we have

$$\int_{-\infty}^{k}\int_{-\infty}^{h} x^2yz\,dx\,dy$$

$$=\int_{-\infty}^{k}\int_{-\infty}^{h}(\phi^2\psi+\psi+2r\phi)\,z\,dx\,dy$$

$$=2!\,2!\,HK\left\{\frac{\theta_1}{2}+\theta_2\theta_1'r+(\theta_{11}T)_1\frac{r^2}{2!}+(\theta_{11}T)_2\frac{r^3}{3!}+\dots\right\}$$

$$+0!\,1!\,HK\left\{1+\theta_1'r+(\theta_{01}T)_1\frac{r^2}{2!}+(\theta_{01}T)_2\frac{r^3}{3!}+\dots\right\}$$

$$+2r1!\,0!\,HK\left\{1+\theta_1r+(\theta_{10}T)_1\frac{r^2}{2!}+(\theta_{10}T)_2\frac{r^3}{3!}+\dots\right\}$$

$$=HK\left[4\left\{\frac{\theta_1}{2}+\theta_2\theta_1'r+(\theta_2\theta_1'T_1T_1'-1)\frac{r^2}{2!}\right.\right.$$

$$\left.+(\theta_2\theta_1'T_2T_2'-2\theta_1T_1T_1')\frac{r^3}{3!}\right.$$

$$\left.+(\theta_2\theta_1'T_2T_2'-3\theta_1T_2T_2')\frac{r^4}{4!}\right\}$$

$$+\left\{1+\theta_1'r+(\theta_1'T_1T_1')\frac{r^2}{2!}+(\theta_1'T_2T_2')\frac{r^3}{3!}+\dots\right\}$$

$$\left.+2r\left\{1+\theta_1r+(\theta_1T_1T_1')\frac{r^2}{2!}+(\theta_1T_2T_2')\frac{r^3}{3!}+\dots\right\}\right] \quad \dots\dots(128).$$

The further simplification of the last two terms is obvious.

12. *The determination of the constants of a whole distribution from those of a quadrant.*

The formulae developed in the preceding pages have many uses, but one of the most useful and interesting is their application to the inverse problem of deducing the constants of a complete normal solid from those of a given quadrant. Thus the records of a body of soldiers might show the age and height distribution, but if recruiting were limited to those over 5′ 8″ in height and over 18 years of age we should have the data of a quadrant only, bounded by a plane at 18 years of age and another at 5′ 8″. The problem is to determine from measurements of this quadrant the constants for the whole population.

If we express the moment coefficients about the general mean, with standard deviations as units, as m_{10}, m_{01}, m_{20}, etc. and write $\frac{N}{m}=\lambda$ we have from the preceding paper

$$m_{10}=-\lambda(HA+rKB),$$
$$m_{01}=-\lambda(rHA+KB),$$
$$m_{20}=-\lambda(hHA-r(1-r^2)\chi+r^2kKB)+1,$$
$$m_{02}=-\lambda(r^2hHA-r(1-r^2)\chi+kKB)+1,$$
$$m_{11}=-\lambda(rhHA-(1-r^2)\chi+rkKB)+r.$$

If now we denote the moments of the quadrant as measured from the bounding planes by q_{10}, q_{01}, q_{20}, etc., with the standard deviations S_x, S_y of the quadrant as units, it can easily be shown that

$$q_{20}S_x^2 = (m_{20} - 2hm_{10} + h^2)\,\sigma_x^2,$$

$$q_{02}S_y^2 = (m_{02} - 2km_{01} + k^2)\,\sigma_y^2,$$

$$q_{11}S_xS_y = (m_{11} - hm_{01} - km_{10} + hk)\,\sigma_x\sigma_y,$$

and

$$Q_{20} = \frac{q_{20}}{q_{10}^2} = \frac{m_{20} - 2hm_{10} + h^2}{(m_{10} - h)^2} = \frac{\lambda\{hHA + r(1-r^2)\chi + r(2h - rk)\,KB\} + 1 + h^2}{\{\lambda(HA + rKB) + h\}^2},$$

$$Q_{02} = \frac{q_{02}}{q_{01}^2} = \frac{m_{02} - 2km_{01} + k^2}{(m_{01} - k)^2} = \frac{\lambda\{kKB + r(1-r^2)\chi + r(2k - rh)\,HA\} + 1 + k^2}{\{\lambda(rHA + KB) + k\}^2},$$

$$Q_{11} = \frac{q_{11}}{q_{10}q_{01}} = \frac{m_{11} - hm_{01} - km_{10} + hk}{(m_{10} - h)(m_{01} - k)} = \frac{\lambda\{hKB + (1-r^2)\chi + kHA\} + r + hk}{\{\lambda(HA + rKB) + h\}\{\lambda(rHA + KB) + k\}}.$$

Q_{20}, Q_{02}, and Q_{11} are known but so far as I can see at present the solution of the above equations would require the construction of four-entry tables in h, k, r, and λ.

If these were found, then

$$\sigma_x = \frac{q_{10}S_x}{-\lambda(HA + rKB) - h},$$

$$\sigma_y = \frac{q_{01}S_y}{-\lambda(rHA + KB) - k},$$

$$m = \frac{N}{\lambda}.$$

In conclusion my thanks are due to Professor Pearson for suggesting the subject of the above enquiry and the interest he has taken in its course, and again to Miss Alison Robertson for assistance in correcting the proofs.

MISCELLANEA.

Table of Ordinates of the Normal Curve for each Permille of Frequency.

EDITORIAL.

The need of a table such as the present has long been felt. A table of the abscissae of the normal curve for every permille of frequency was calculated by Sheppard at Galton's suggestion*, but the corresponding table for the ordinates to five significant figures, which is equally needful, has not hitherto been provided.

When we have data classified in broad categories, whether we are dealing with one or two variates it is most desirable to exhibit the results in some graphical form in order that the lazy reader may have an appreciation of the matter under discussion. In the present state of our knowledge, whatever more or less justifiable criticisms may be raised, we do learn something by exhibiting the data with the marginal frequencies represented on the normal scale. The main answer to such criticisms is that such a representation is better than none at all. It will not give an association of the wrong sense, and when the correlation is at all sensible will for many practical purposes indicate its drift. The chief source of error lies, of course, when the number of categories is few, in treating the arrays of a second variable for a broad category of the first as if these arrays were themselves normal distributions. Even if the whole distribution were truly normal this would not be the case, but only the case approximately if the categories ceased to be 'broad,' and were replaced by small subranges. In ordinary practice the chief difficulty of the assumption arises from the fact that the centroid of the whole system for the second variate may not be in adequate numerical agreement with the centroid of the centroids of the arrays obtained on the normal assumption. It is true that if we deal with the whole system as normal and confine ourselves to a tripartite division of the arrays and marginal totals of the second variate we can by aid of the tetrachoric functions obtain better appreciations to the centroids of the arrays; not only does this involve a previous knowledge of the correlation coefficient, but the whole process, especially if it has to be applied to a large number of contingency tables, is very laborious. We are in using it in fact overlooking the main point that all we need is a rough diagrammatic exhibition of the general drift of the association. Those who have had occasion to plot large numbers of contingency tables to normal scales will be the first to admit the weaknesses of the method, but the last to assert that such diagrams are without value. They indicate quite effectively to the casual reader that here the association is of no practical importance, and that there it is an essential feature of the characters under investigation.

In reducing data given in broad categories to a normal scale, all abscissae (x) are measured in terms of the unknown standard deviation, all ordinates (z) in terms of unit frequency and unit standard deviation, i.e. $z = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.

Now it is well known that the abscissa of the centroid $\bar{x}_{ss'}$ of a broad category lying between x_s and $x_{s'}$ is numerically given by

$$\bar{x}_{ss'} = (z_{s'} - z_s)/n_{ss'},$$

where $n_{ss'}$ represents the proportional frequency between x_s and $x_{s'}$. In most cases it will be, for diagram purposes, quite adequate to obtain the abscissae and ordinates of the normal scale from the proportional frequencies to three places of decimals. Hence Table I of *Tables for Statisticians* and the present table give the required values in a few minutes, while the old process of determining z from Table II or, where it permitted, Table III was much more lengthy.

Illustration. Find the boundaries and means of the following system of broad categories on the assumption that it corresponds to a normal frequency distribution†.

* *Biometrika*, Vol. v. p. 405. It is reissued as Table I of the *Tables for Statisticians and Biometricians*.

† The continuous variate may be looked upon as physiological fitness for life.

Health of Yearling Male Babies.

		Frequency	Permille	Differences
I.	Very satisfactory	54	·038	·038
II.	Satisfactory	326	·265	·227
III.	Normal	508	·618	·353
IV.	Indifferent	129	·708	·090
V.	Unsatisfactory	198	·846	·138
VI.	Dead	221	1·000	·154
	Total	1436		

The permilles are obtained by a continuous process with the reciprocal of 1436 on the machine. The differences are obtained from the permilles by subtraction and only differ in one case, that of the Normal Health, which if found directly from the frequency would be ·354, and this is of no importance for our present purposes. Table I and the present Table provide at once :

	Abscissa	Ordinate	Difference of Ordinate	Centroid
	$+\infty$	0		
I.	... ...		+·08265	+2·1750
	+1·7744	·08265		
II.	... ...		+·24489	+1·0788
	+0·6280	·32754		
III.	... ...		+·05382	+ ·1525
	+0·3002	·38136		
IV.	... ...		−·03795	− ·4217
	−0·5476	·34341		
V.	... ...		−·10614	− ·7691
	−1·0194	·23727		
VI.	... ...		−·23727	−1·5407
	$-\infty$	0		

The whole work is very simple, as there is no trouble of interpolation at all.

In actually computing the present table of ordinates to permilles a different process was needful for the first eleven rows from that for the remainder of the rows. From ·110 to ·880 it was found adequate to take the x's from Table III of the *Tables for Statisticians* which are given for a for every ·01 of value—and therefore for every ·005 of frequency—and interpolate by Everett's Central-Difference Formula to every ·001 of frequency. It was also sufficient to use only the δ^2 terms. Such a method was not possible for the upper part of the table. Here up to ·02, the value of x was determined by inverse summation from the values in Table II of the *Tables for Statisticians*. Then z was actually computed from $z=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$ for the columns under the rubrics ·000, ·005 and ·010. From these columns the remaining column values were determined using Everett's Central-Difference Formula and including δ^4 terms where requisite. The three rows at the top of the table were found by determining x by inverse summation as before and calculating each individual value of z from its x; the first fifth of the table was thus far more laborious than the last four-fifths. The values of z were ultimately cut down to five figures. A superscript + or − is attached to every value of z ending in 5 for the aid of those who wish to use the table to four figures only, which is often adequate.

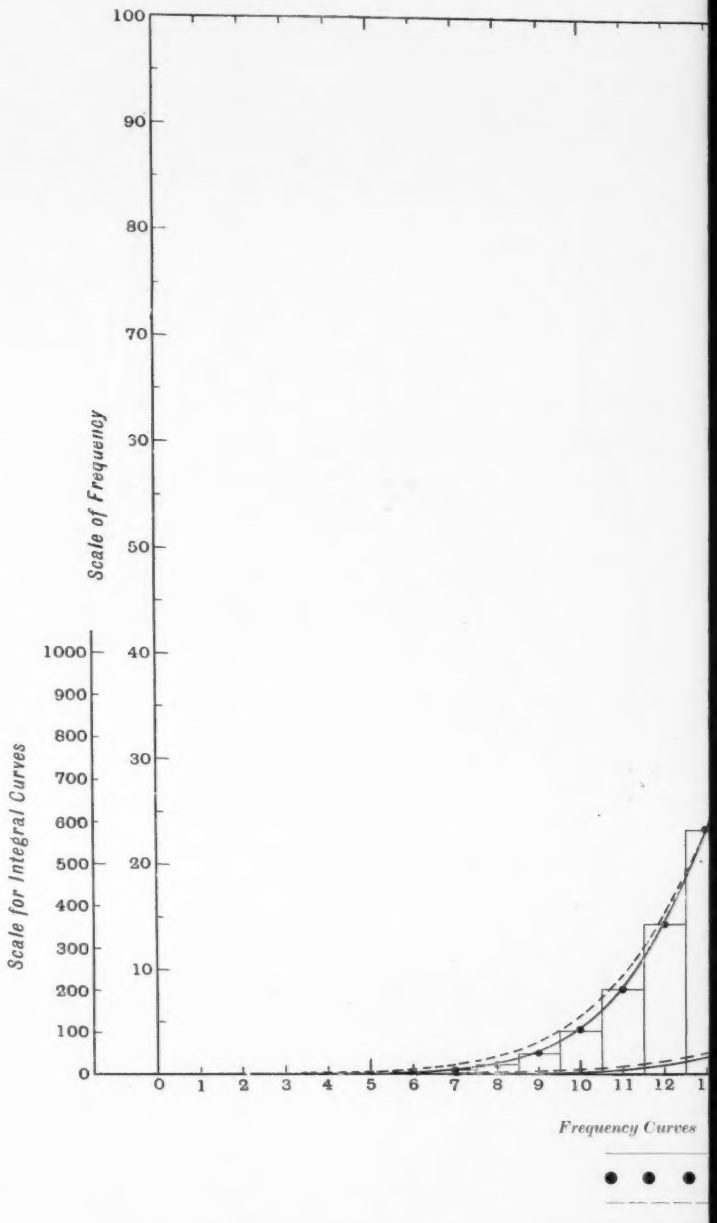
We owe a careful revision of the table to Mr H. E. Soper, who used a much simplified process and modified the last figure in a number of cases. If m be the permille, i.e. $\frac{1}{2}(1+a)$, x the abscissa and z the ordinate, he took the formula

$$\delta z = x\delta m - \frac{(\delta m)^2}{2z} + \frac{(\delta m)^3}{6z^2} - \ldots,$$

which enabled him to start from the z of the nearest tabled m in Table II of the *Tables for Statisticians*.

Table of Ordinates of the Normal Curve for each Permille of Frequency.

Permille	·000	·001	·002	·003	·004	·005	·006	·007	·008	·009	·010	
·00	·00000	·00337	·00634	·00915⁻	·01185⁻	·01446	·01700	·01949	·02192	·02431	·02665⁺	·99
·01	·02665⁺	·02896	·03123	·03348	·03568	·03787	·04003	·04216	·04427	·04635⁺	·04842	·98
·02	·04842	·05046	·05249	·05449	·05648	·05845⁺	·06040	·06233	·06425⁺	·06615⁺	·06804	·97
·03	·06804	·06992	·07177	·07362	·07545⁺	·07727	·07908	·08087	·08265⁻	·08442	·08617	·96
·04	·08617	·08792	·08965⁺	·09137	·09309	·09479	·09648	·09816	·09983	·10149	·10314	·95
·05	·10314	·10478	·10641	·10803	·10964	·11124	·11284	·11442	·11600	·11756	·11912	·94
·06	·11912	·12067	·12222	·12375⁻	·12528	·12679	·12830	·12981	·13130	·13279	·13427	·93
·07	·13427	·13574	·13720	·13866	·14011	·14156	·14299	·14442	·14584	·14726	·14867	·92
·08	·14867	·15007	·15146	·15285⁺	·15423	·15561	·15698	·15834	·15970	·16105⁻	·16239	·91
·09	·16239	·16373	·16506	·16639	·16770	·16902	·17033	·17163	·17292	·17421	·17550	·90
·10	·17550	·17678	·17805⁻	·17932	·18058	·18184	·18309	·18433	·18557	·18681	·18804	·89
·11	·18804	·18926	·19048	·19169	·19290	·19410	·19530	·19649	·19768	·19886	·20004	·88
·12	·20004	·20121	·20238	·20354	·20470	·20585⁺	·20700	·20814	·20928	·21042	·21155⁻	·87
·13	·21155⁻	·21267	·21379	·21490	·21601	·21712	·21822	·21932	·22041	·22149	·22258	·86
·14	·22258	·22365⁺	·22473	·22580	·22686	·22792	·22898	·23003	·23108	·23212	·23316	·85
·15	·23316	·23419	·23522	·23625⁻	·23727	·23829	·23930	·24031	·24131	·24232	·24331	·84
·16	·24331	·24430	·24529	·24627	·24726	·24823	·24921	·25017	·25114	·25210	·25305⁺	·83
·17	·25305⁺	·25401	·25495⁺	·25590	·25684	·25778	·25871	·25964	·26056	·26148	·26240	·82
·18	·26240	·26331	·26422	·26513	·26603	·26693	·26782	·26871	·26960	·27049	·27137	·81
·19	·27137	·27224	·27311	·27398	·27485⁻	·27571	·27657	·27742	·27827	·27912	·27996	·80
·20	·27996	·28080	·28164	·28247	·28330	·28413	·28495⁻	·28577	·28658	·28739	·28820	·79
·21	·28820	·28901	·28981	·29060	·29140	·29219	·29298	·29376	·29454	·29532	·29609	·78
·22	·29609	·29686	·29763	·29840	·29916	·29991	·30067	·30142	·30216	·30291	·30365⁻	·77
·23	·30365⁻	·30439	·30512	·30585⁻	·30658	·30730	·30802	·30874	·30945⁺	·31017	·31087	·76
·24	·31087	·31158	·31228	·31298	·31367	·31436	·31505⁺	·31574	·31642	·31710	·31778	·75
·25	·31778	·31845⁻	·31912	·31979	·32045⁻	·32111	·32177	·32242	·32307	·32372	·32437	·74
·26	·32437	·32501	·32565⁻	·32628	·32691	·32754	·32817	·32879	·32941	·33003	·33065⁻	·73
·27	·33065⁻	·33126	·33187	·33247	·33307	·33367	·33427	·33486	·33545⁻	·33604	·33662	·72
·28	·33662	·33720	·33778	·33836	·33893	·33950	·34007	·34063	·34119	·34175⁻	·34230	·71
·29	·34230	·34286	·34341	·34395⁺	·34449	·34503	·34557	·34611	·34664	·34717	·34769	·70
·30	·34769	·34822	·34874	·34925⁺	·34977	·35028	·35079	·35129	·35180	·35230	·35279	·69
·31	·35279	·35329	·35378	·35427	·35475⁺	·35524	·35572	·35620	·35667	·35714	·35761	·68
·32	·35761	·35808	·35854	·35900	·35946	·35991	·36037	·36082	·36126	·36171	·36215⁻	·67
·33	·36215⁻	·36259	·36302	·36346	·36389	·36431	·36474	·36516	·36558	·36600	·36641	·66
·34	·36641	·36682	·36723	·36764	·36804	·36844	·36884	·36923	·36962	·37001	·37040	·65
·35	·37040	·37078	·37116	·37154	·37192	·37229	·37266	·37303	·37340	·37376	·37412	·64
·36	·37412	·37447	·37483	·37518	·37553	·37588	·37622	·37656	·37690	·37724	·37757	·63
·37	·37757	·37790	·37823	·37855⁺	·37888	·37920	·37951	·37983	·38014	·38045⁻	·38076	·62
·38	·38076	·38106	·38136	·38166	·38196	·38225⁺	·38254	·38283	·38312	·38340	·38368	·61
·39	·38368	·38396	·38423	·38451	·38478	·38504	·38531	·38557	·38583	·38609	·38634	·60
·40	·38634	·38659	·38684	·38709	·38734	·38758	·38782	·38805⁺	·38829	·38852	·38875⁻	·59
·41	·38875⁻	·38897	·38920	·38942	·38964	·38986	·39007	·39028	·39048	·39069	·39089	·58
·42	·39089	·39109	·39129	·39149	·39168	·39187	·39206	·39224	·39243	·39261	·39279	·57
·43	·39279	·39296	·39313	·39330	·39347	·39364	·39380	·39396	·39411	·39427	·39442	·56
·44	·39442	·39457	·39472	·39486	·39501	·39514	·39528	·39542	·39555⁻	·39568	·39580	·55
·45	·39580	·39593	·39605⁺	·39617	·39629	·39640	·39651	·39662	·39673	·39683	·39694	·54
·46	·39694	·39703	·39713	·39723	·39732	·39741	·39749	·39758	·39766	·39774	·39781	·53
·47	·39781	·39789	·39796	·39803	·39809	·39816	·39822	·39828	·39834	·39839	·39844	·52
·48	·39844	·39849	·39854	·39858	·39862	·39866	·39870	·39873	·39876	·39879	·39882	·51
·49	·39882	·39884	·39886	·39888	·39890	·39891	·39892	·39893	·39894	·39894	·39894	·50
	·010	·009	·008	·007	·006	·005	·004	·003	·002	·001	·000	Permille


Scale of Frequency
Scale for Integral Curves
100
90
80
70
30
50
40
30
20
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900
800
700
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500
400
300
200
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0
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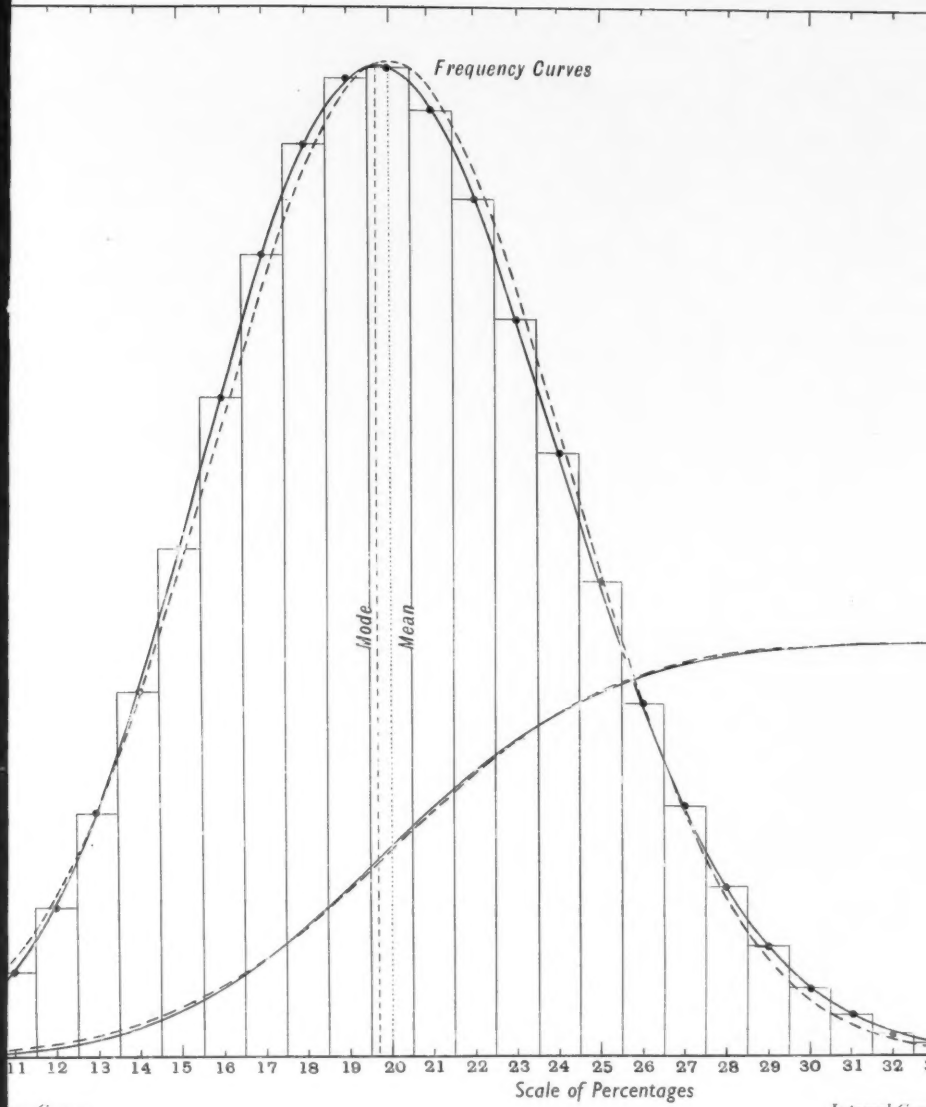
Frequency Curves


Frequency Curves
Mode
Mean
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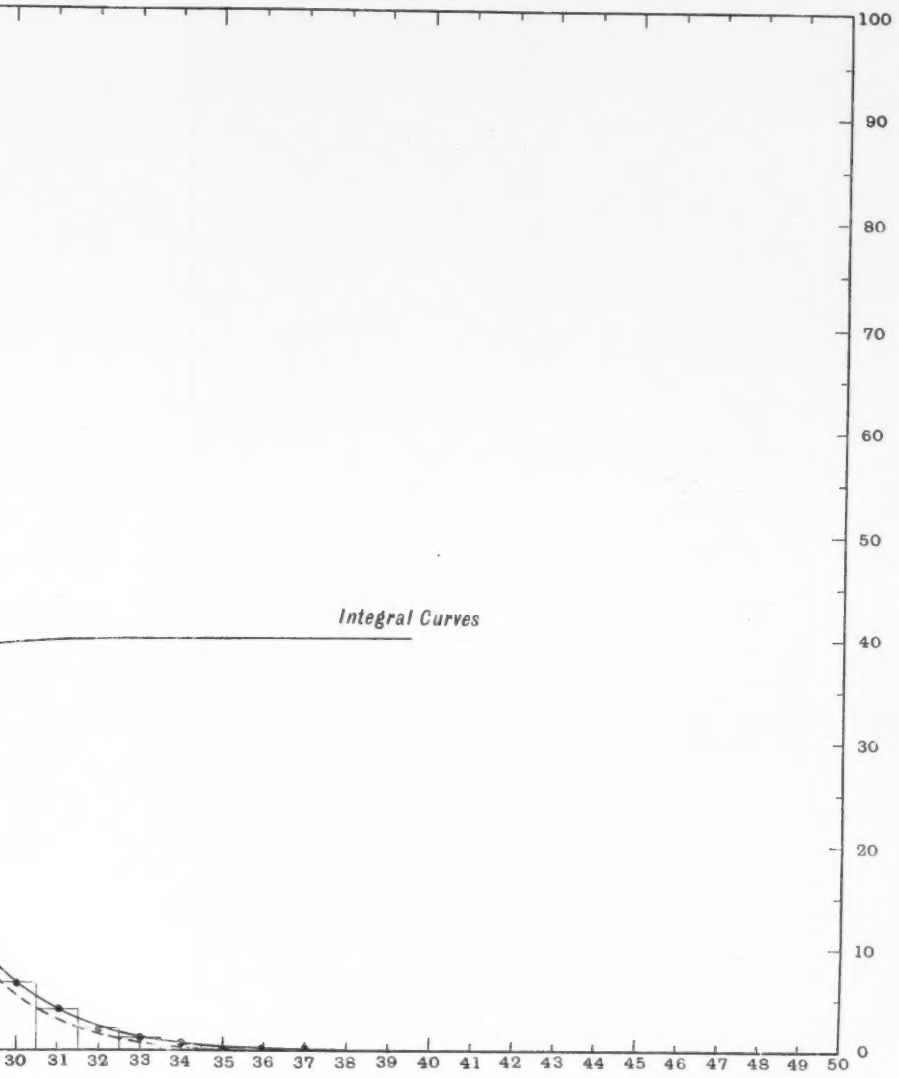
Scale of Percentages
ncy Curves
Integral Cur
Skew Curve from Geometrical Relation
Histogram Points
Normal Curve

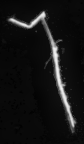
Diagram I

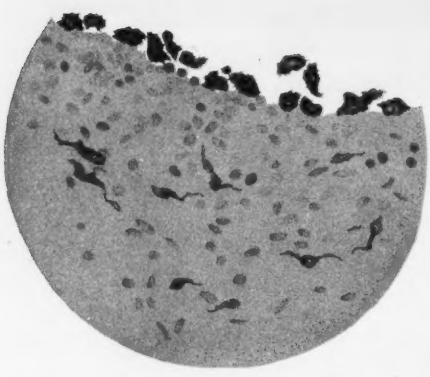


Integral Curves

Histogram

Normal Curve





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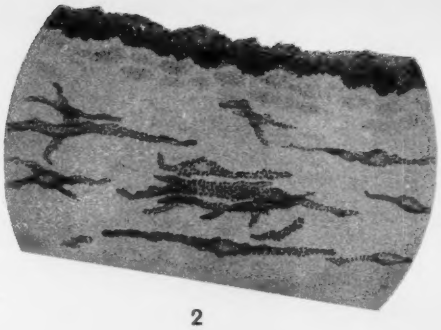

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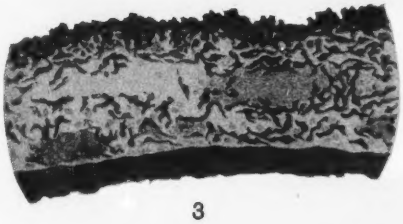

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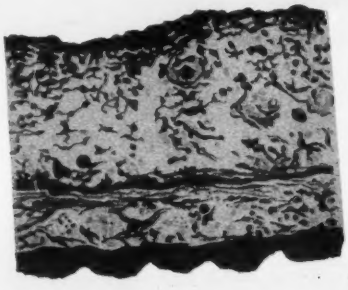

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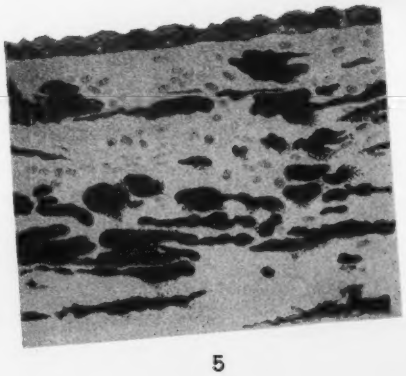
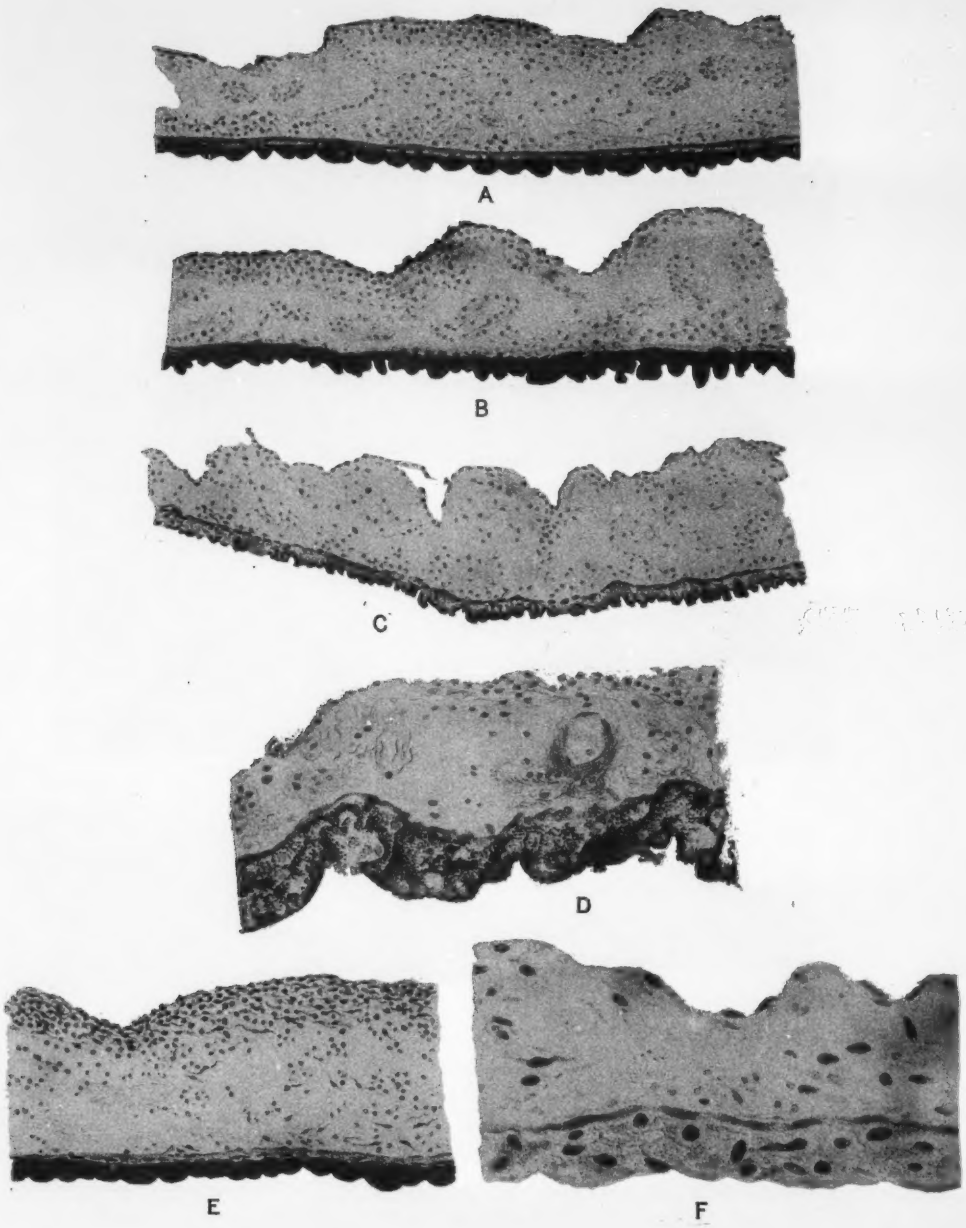


5

1, 2, 3, Tamil foetus at different ages; 4, 5, Tamil adults.



A, C, D and F, British Albinos; E, Negro Albino.

Fig. 1

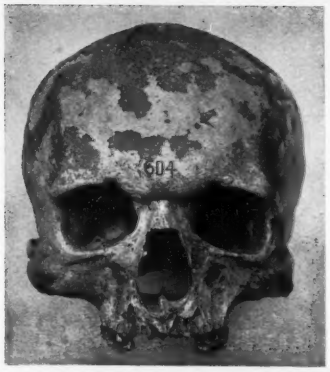


Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Modern and Ancient Greek Heads.

For Description, see p. 112.

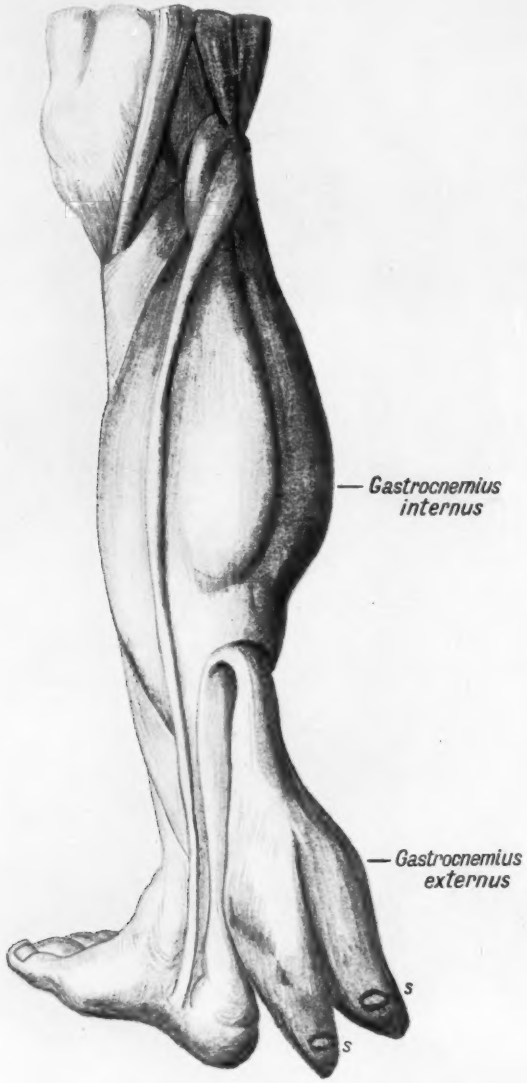


Fig. 1. First attempt at illustration of the *fabellae* in the two heads of *gastrocnemius*. From Casserius' *Tabulae anatomicae*, 1632.

Fig. 2. (c. nat. size.) Sesamoids of first metatarsal of foot in man. Sesamoids known to the Greeks. After Pfitzner.

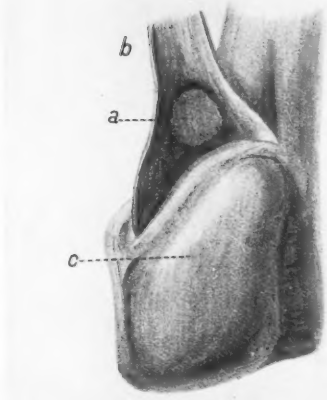


Fig. 3. *Fabella lateralis* as figured by Gillette, 1872. See description of plates.

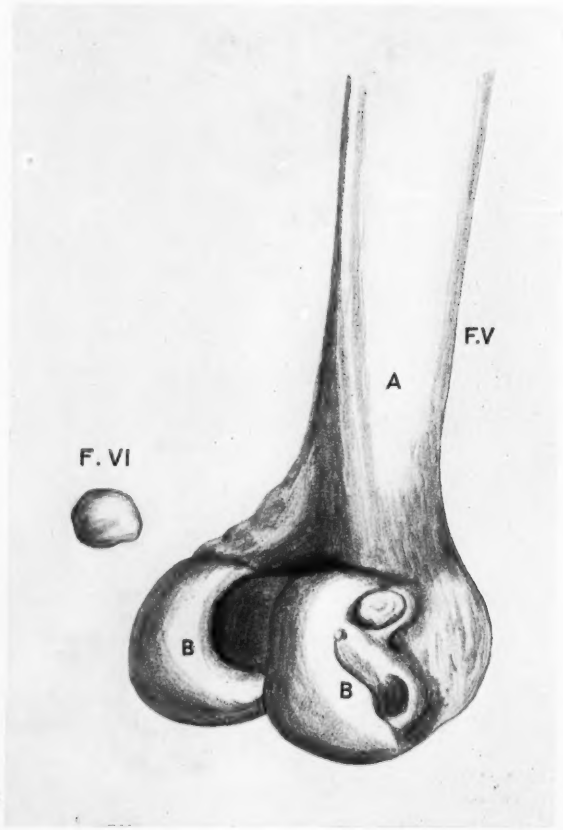


FIG. 4. Trew's rediscovery of the *fabella lateralis*, 1715. Note the unusual position given to sesamoid *C*. (The letter *C* should have been inserted above right hand *B*.)

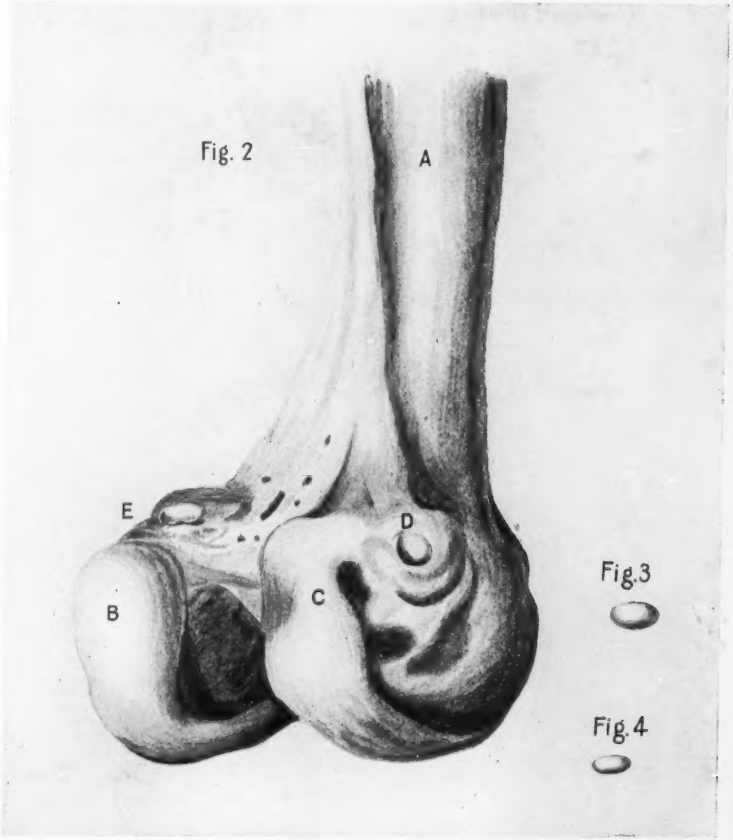


FIG. 5. Heister's "improvement" on Trew; he provided a small *fabella mesialis* in 1732. Both his sesamoids are in very unusual if not impossible places. See description of plates.

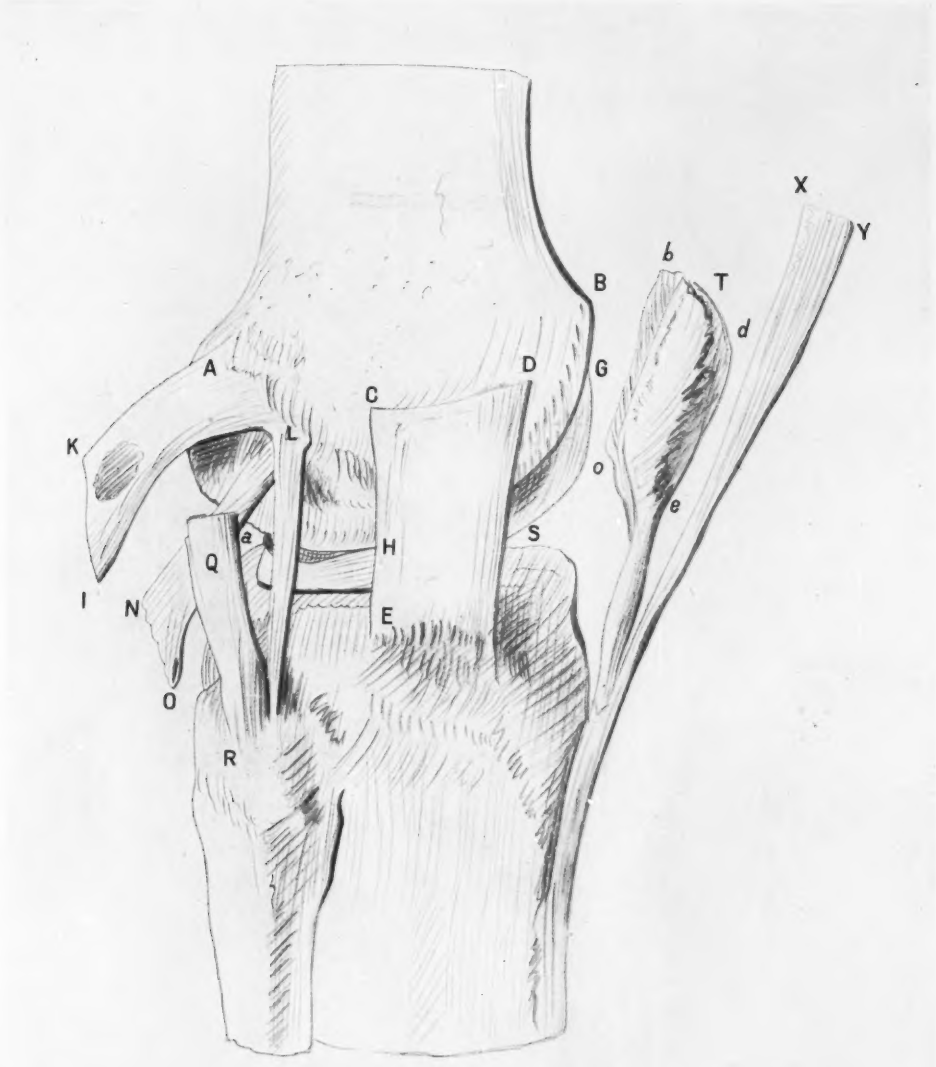


FIG. 6. (c. nat. size.) Peter Camper's illustration of the external *fabella* (near *K*) in the tendon of *gastrocnemius*, 1754.

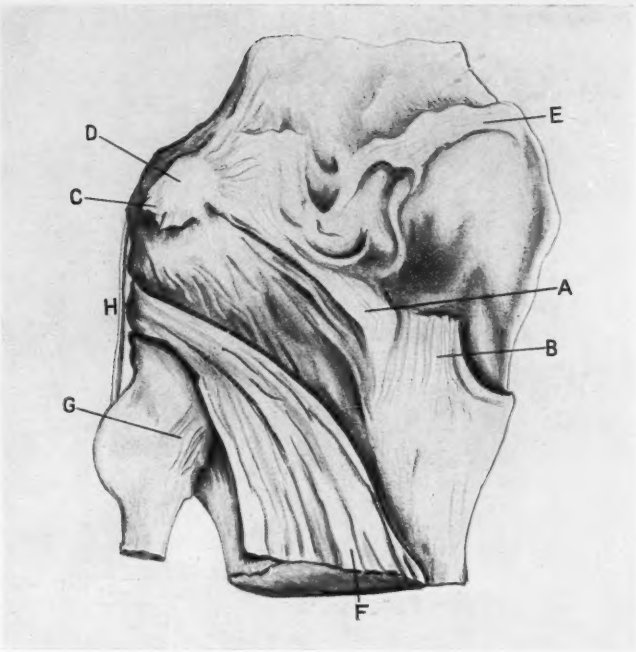


FIG. 7. (c. $\frac{1}{4}$ nat. size.) Sir George Humphry's illustration of the *fabella lateralis*. The sesamoid in correct position, 1858. See description of plates.

FIG. 8. (c. $\frac{1}{2}$ nat. size.) Pathological ossification at head of internal *gastrocnemius*, Gruber, 1876.

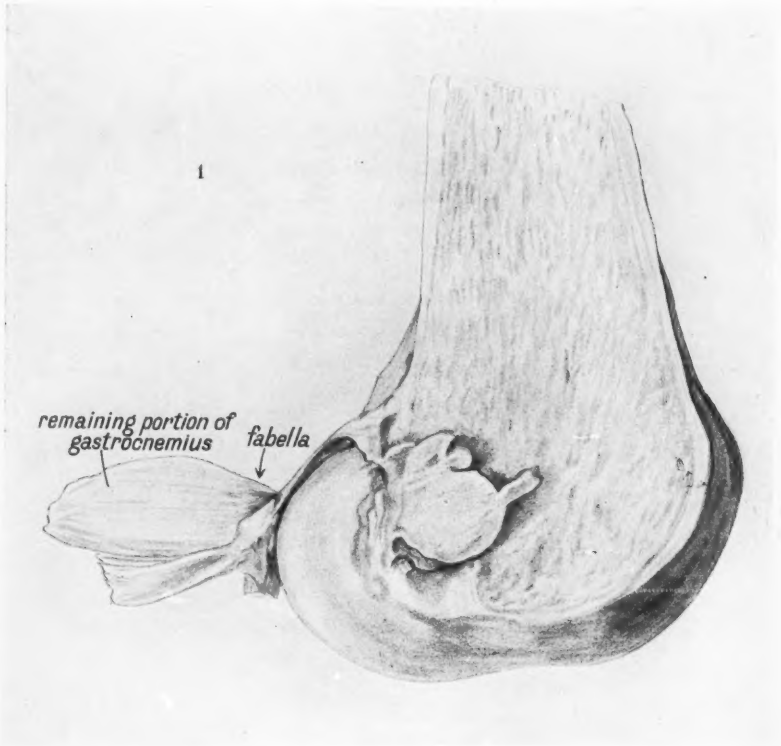
FIG. 9. (c. $\frac{1}{2}$ nat. size.) *M. popliteus biceps*, the additional head as well as the *ligamentum posticum* and *M. gastrocnemius* joining the *fabella lateralis*, Gruber, 1876.

FIG. 10. (c. $\frac{1}{2}$ nat. size.) Ossicle (4) at "Knotenpunkt" of *M. gastrocnemius* (*n*), *Ligamentum popliteum* (*b*), and *Ligamentum posticum* (*d*), Gruber, 1876.

FIG. 13. (c. nat. size.) Sesamoid on the head of the fibula in the case of the Wombat, Pfitzner, 1892. See description of plates.

FIG. 11. (c. $\frac{3}{4}$ nat. size.) External *fabella* (4) in relation to *M. gastrocnemius* (*k*), and *M. plantaris* (*l*), Gruber, 1876.

FIG. 12. (c. $\frac{3}{4}$ nat. size.) External *fabella* (4) in relation to *M. gastrocnemius* (*k*), *Ligamentum popliteum* (*b*), and *Ligamentum posticum* (*d*), Gruber, 1876.



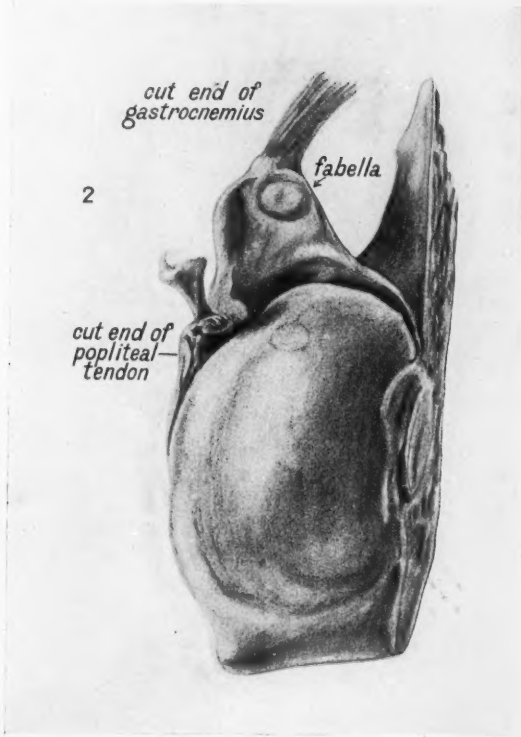


FIG. 14

FIG. 15

Drawings illustrating the position of the external *fabella* in man from a case occurring in the Dissecting-Room at University College, 1919. This is a relatively rare position: compare the skiagrams of the knee-joint. See description of plates. Circa $\frac{5}{4}$ nat. size.

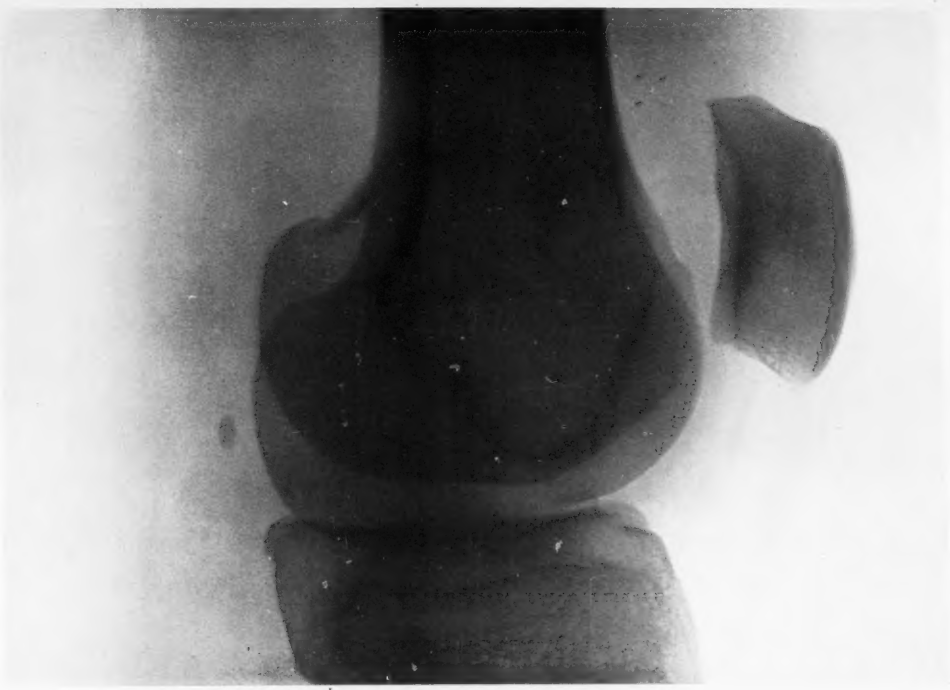


FIG. 16. Small lateral *fabella*.

FIG. 17. Large lateral *fabella*.

FIG. 18. Large lateral *fabella*.

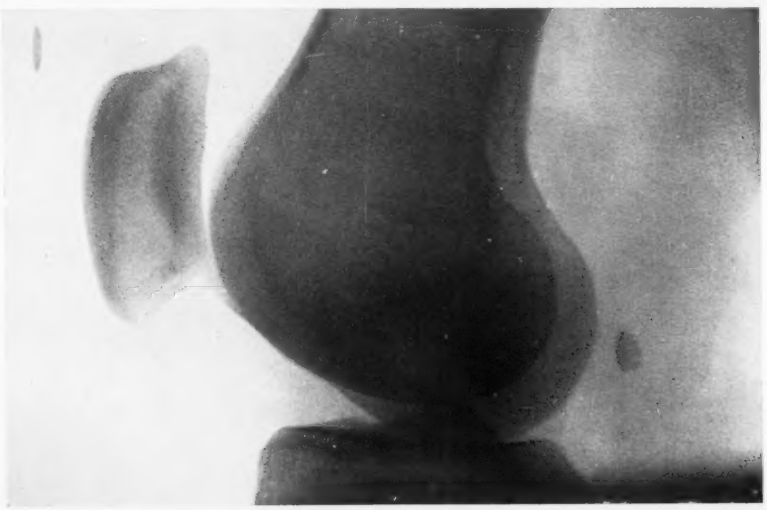


FIG. 19. Lateral *fabella*.

FIG. 20. Small lateral *fabella*.

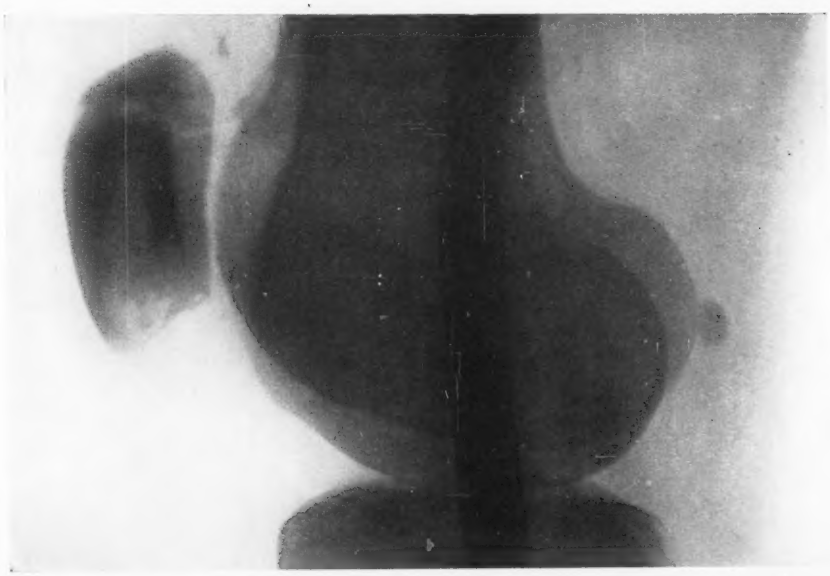


FIG. 21. Large lateral *fabella*.

FIG. 22. Lateral *cyamella* in popliteal sulcus in posterior aspect.

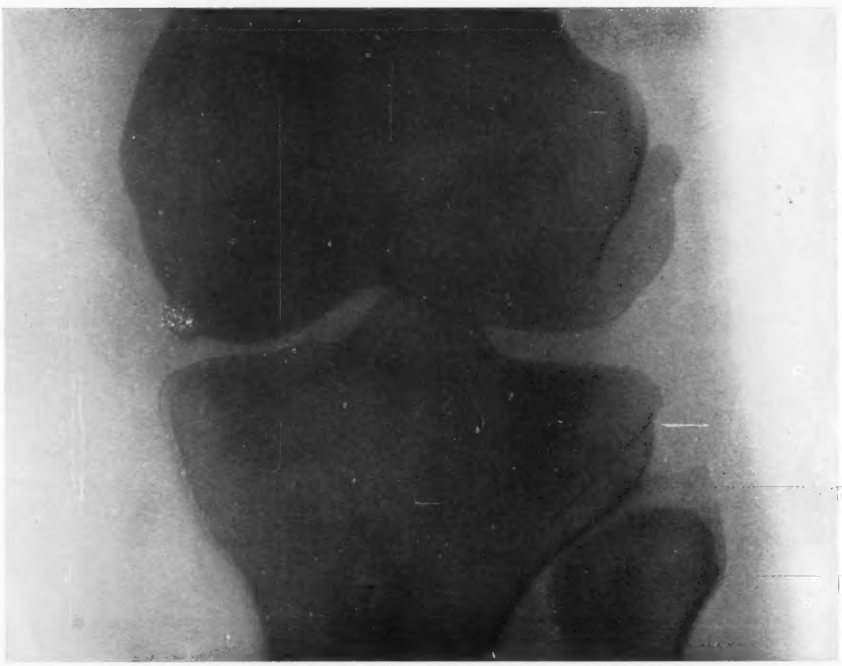


FIG. 23. The same sesamoid in anterior aspect.
Note in both figures the possibility of a mesial *cyamella*.

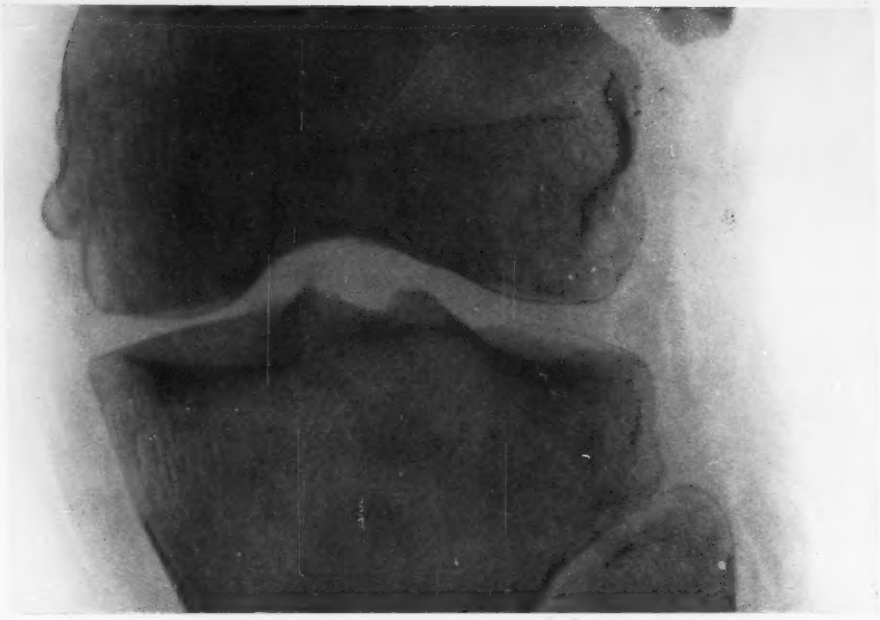


FIG. 24. Occurrence of what might possibly be interpreted as a mesial *cyamella*.

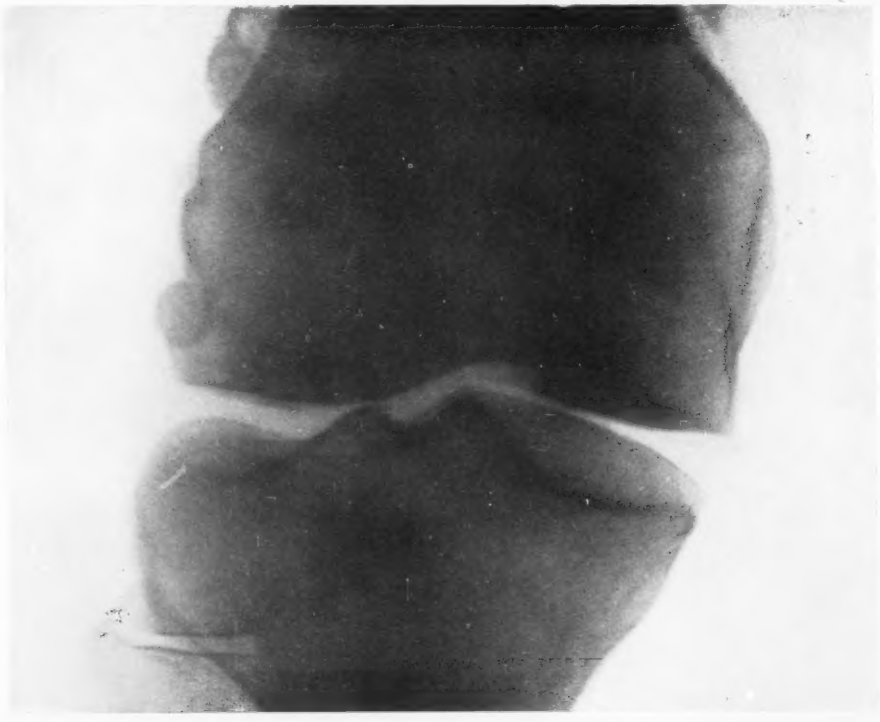


FIG. 25. Lateral *cyamella* in popliteal sulcus.

F

FIG. 26. Lateral *lunula*.

FIG. 27. Another case of lateral *lunula*. Also a sesamoid above the head of the fibula (? displaced *fabella*.)

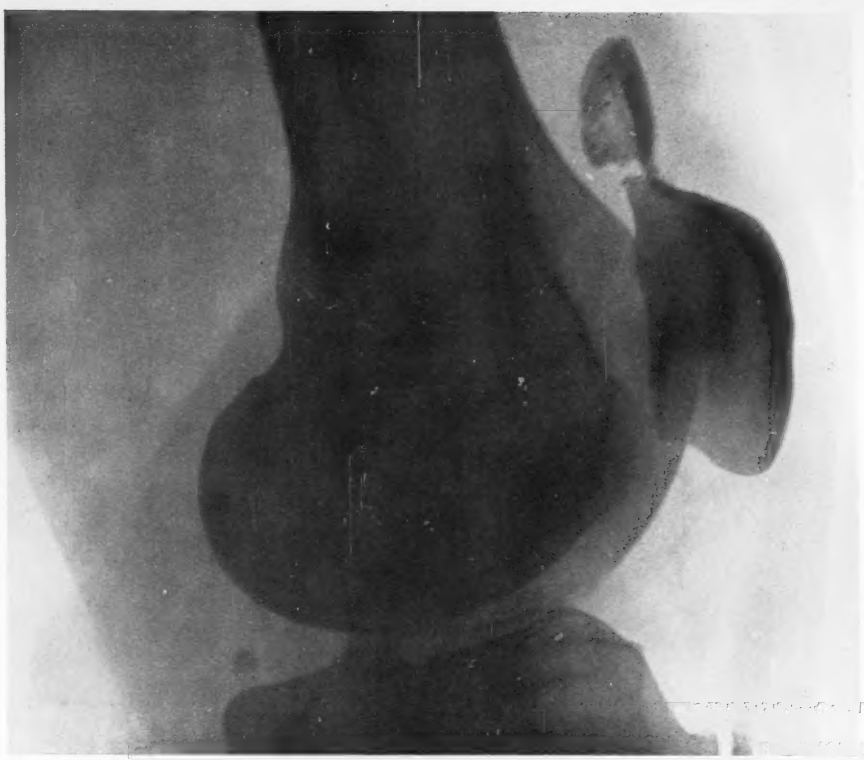


FIG. 28. Double *patella* and posterior *lunula* (?).

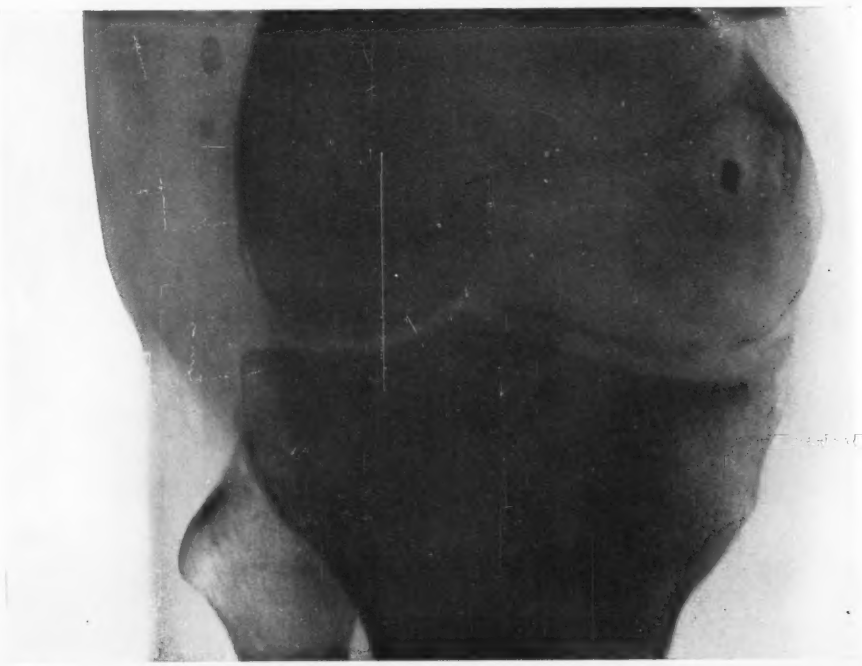


FIG. 29. Two small lateral sesamoids in unusual positions, possibly associated with the lateral femoro-fibular ligament.

FIG. 30. Double *patella*. (? Small external posterior *lunula*.)

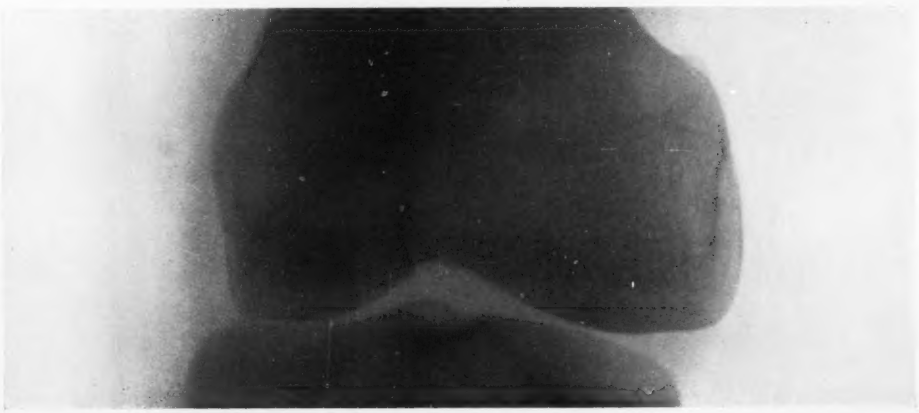


FIG. 31. Sesamoid in femoro-fibular ligament? Cf. Plate XIII, Fig. 29.

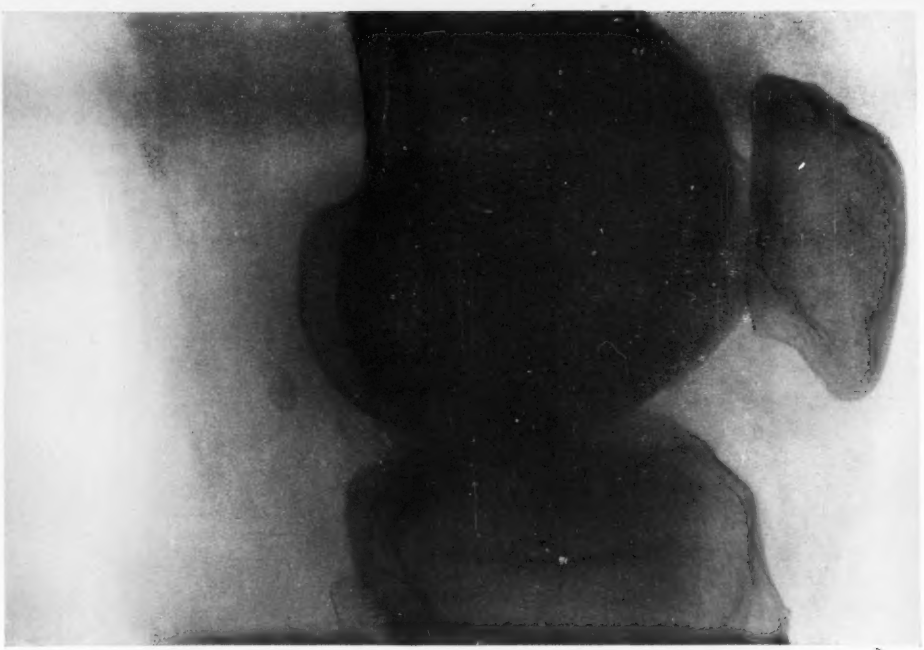


FIG. 32. Lateral *fabella*.

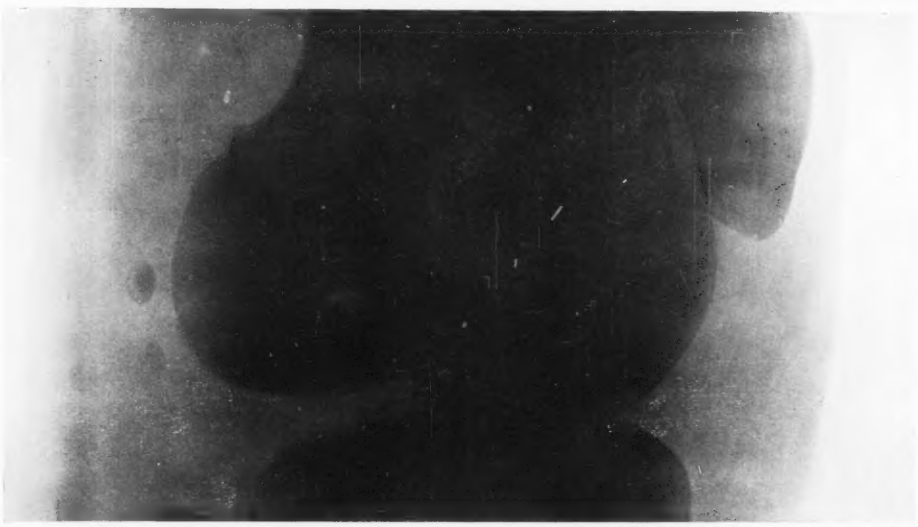


FIG. 33. Lateral *fabella*, with some trace of a second sesamoid beneath it.

FIGS. 32 and 33 are reproduced from Atlas, Part II, to the *Monograph on The Long Bones of the English Skeleton*, by Pearson and Bell.

FIG. 34. Lateral *fabella*.

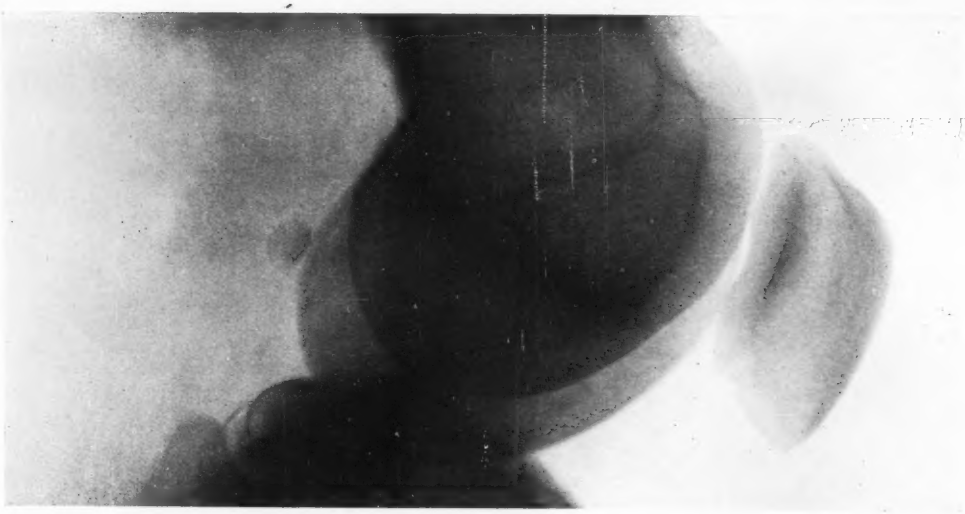


FIG. 35. Lateral *fabella*.

FIGS. 34, 35 are reproduced from Atlas, Part II, to the *Monograph on The Long Bones of the English Skeleton*, by Pearson and Bell.

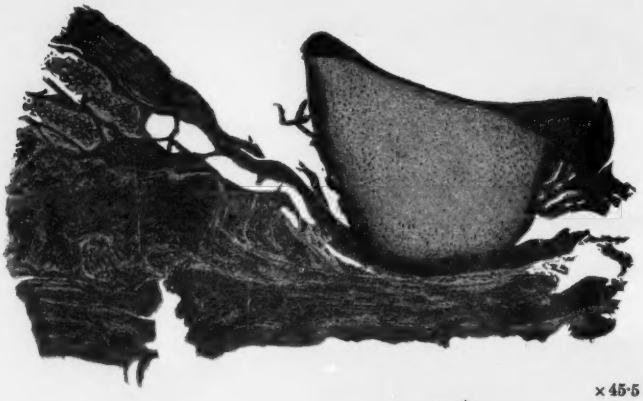


Fig. 36 (i). ×45·5

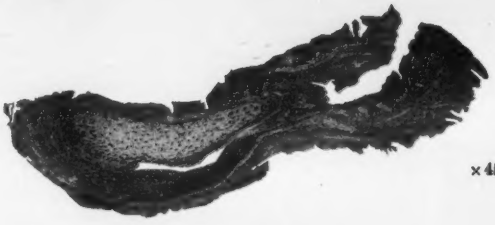
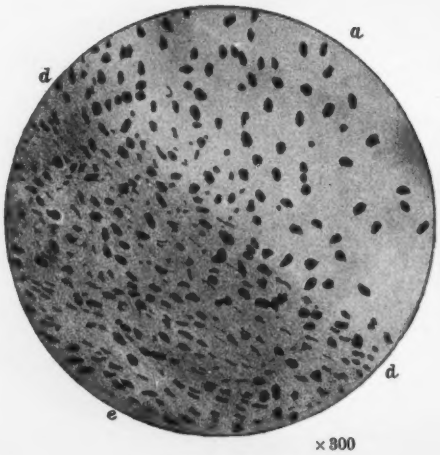
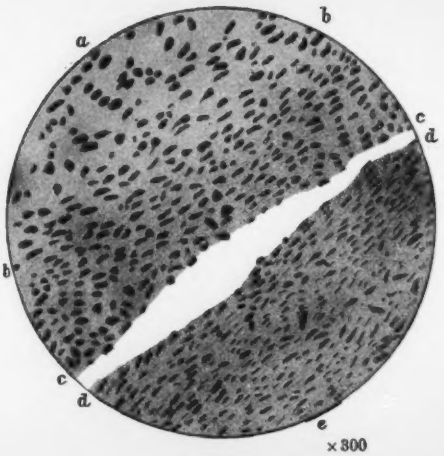


Fig. 37 (i). ×45·5



×300

Fig. 36 (ii).



×300

Fig. 37 (ii).

Sections of the Sesamoids of a Newly-born Kitten.

Fig. 38 (i). ×45·5

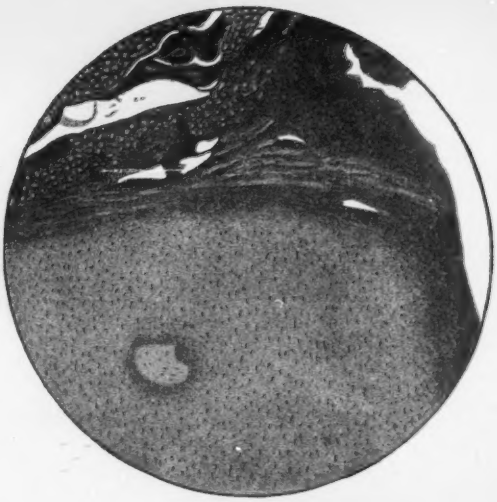


Fig. 39 (i). ×45·5

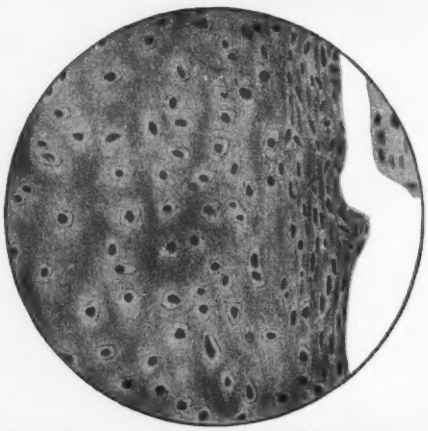


Fig. 38 (ii). ×300

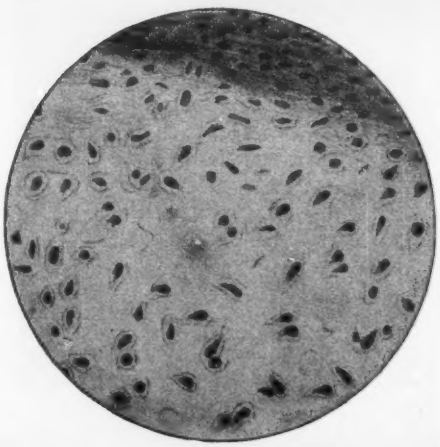


Fig. 39 (ii). ×300

Sections of the Sesamoids of a full-term Human Foetus.

FIG. 1

FIG. 2

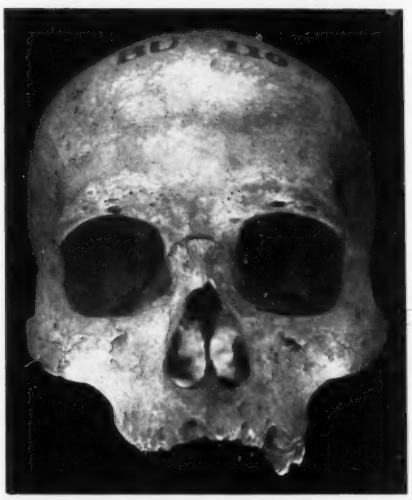


FIG. 3

Cranium of Typical ♂ Burman, No. 110.

FIG. 1. N. lateralis. FIG. 2. N. verticalis. FIG. 3. N. facialis.

FIG. 4

FIG. 5

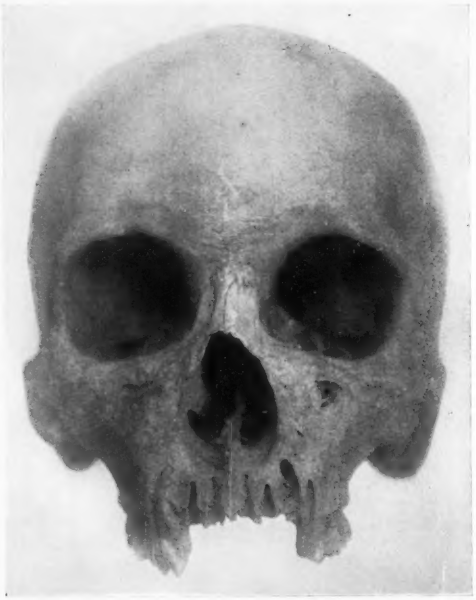


FIG. 6

FIG. 7

Cranium of Typical ♂ Burman, No. 4.

FIG. 4. N. occipitalis. Fig. 5. N. verticalis. FIG. 6. N. facialis. FIG. 7. N. basalis.

FIG. 8

Typical ♂ Burman, No. 4. FIG. 8. N. lateralis.

FIG. 9

FIG. 10

Typical ♂, Type C (Karen ?).

FIG. 9. N. lateralis. FIG. 10. N. facialis.

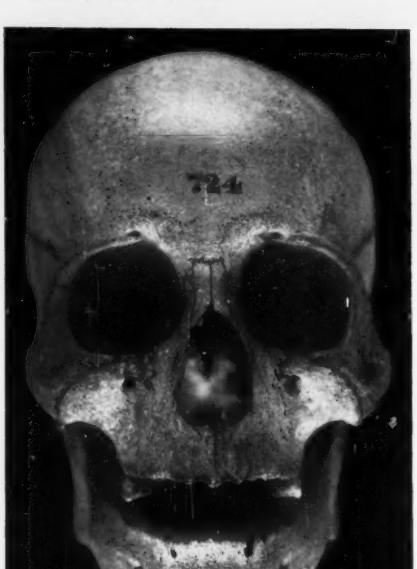


FIG. 11

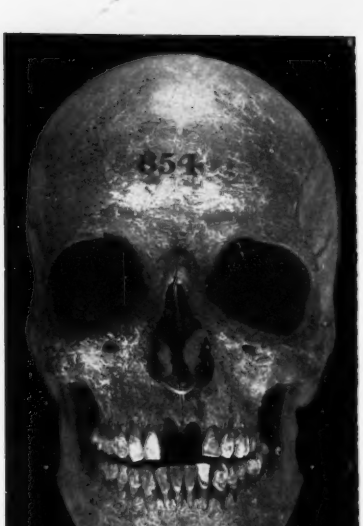


FIG. 13

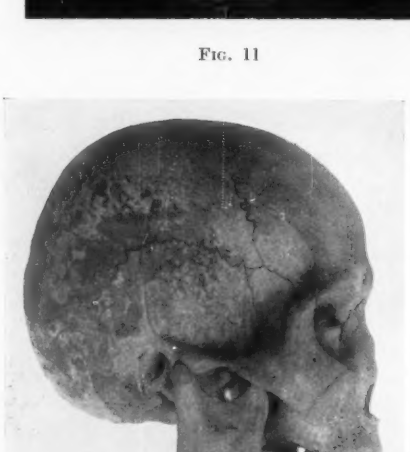


FIG. 12

FIG. 14

Karen ♂ Skull, No. 724.
Fig. 11. N. facialis. Fig. 12. N. lateralis.

Maravar ♂ Skull, No. 654.
Fig. 13. N. facialis. Fig. 14. N. lateralis.

From Crania in the Museum of the Royal College of Surgeons.

FIG. 15

FIG. 16

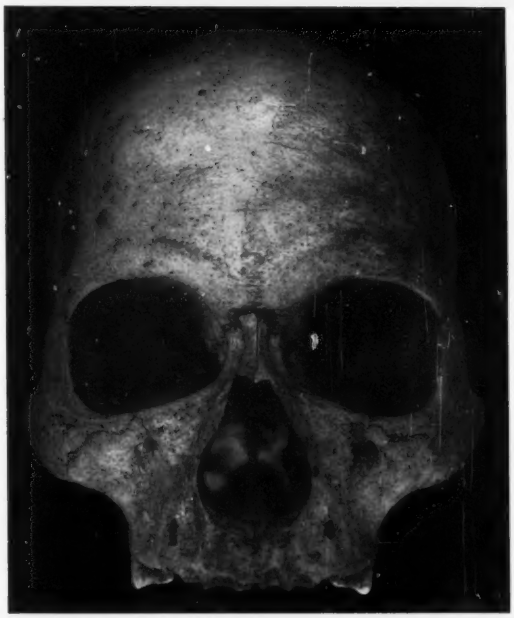


FIG. 17

Typical ♂ Hindu (Lower Bengal). No. 3 of Hindu Series in Biometric Laboratory.
Fig. 15. N. lateralis. Fig. 16. N. verticalis. Fig. 17. N. facialis.

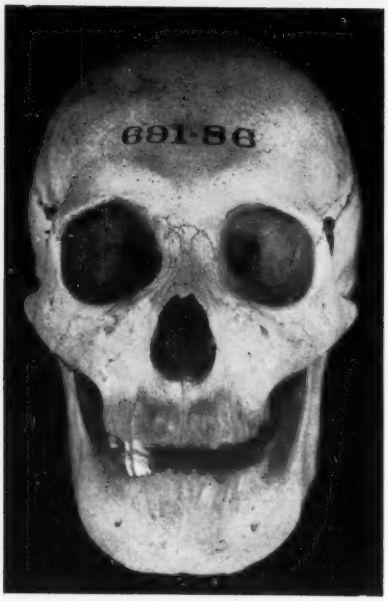


FIG. 18

FIG. 19

Young ♂ Chinaman.

Fig. 18. N. facialis. Fig. 19. N. lateralis.

FIG. 20

FIG. 21

Typical ♂ Malay.

Fig. 20. N. facialis. Fig. 21. N. lateralis.

Fig.

FIG. 22 (No. 23).

FIG. 23 (No. 136).

FIG. 24 (No. 95).

FIG. 25 (No. 13).

Special Burmese Crania, illustrating abnormalities.

Fig. 22. Right temporal bone divided into three separate parts. Fig. 23. Peculiar ossification of temporal suture.
Fig. 24. Two pre-condyles fused. Fig. 25. Two pre-condyles.

Fig. 2

FIG. 26 (No. 113).

FIG. 27 (No. 82).

FIG. 28 (No. 8).

FIG. 29 (No. 92).

Special Burmese Crania, illustrating abnormalities.

Fig. 26. Very deep canine fossae. Fig. 27. *Os japonicum* on left.
Fig. 28. Left upper canine thrust in lateral direction. Fig. 29. Unusual fossa at anterior border of *foramen magnum*.

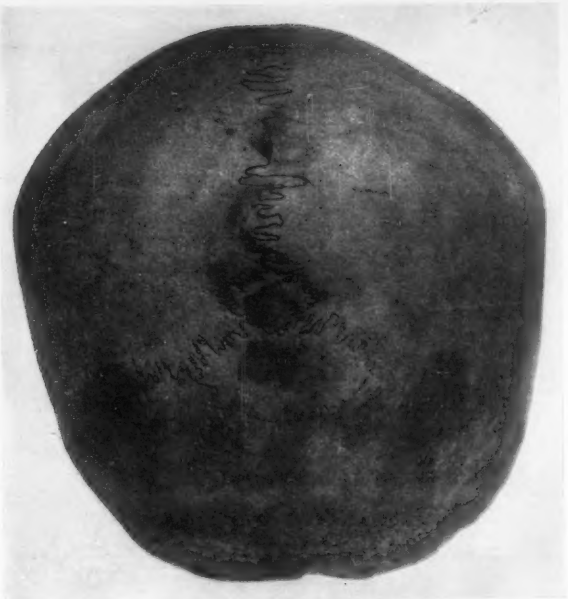


FIG. 30 (No. 121).

FIG. 31 (No. 74).

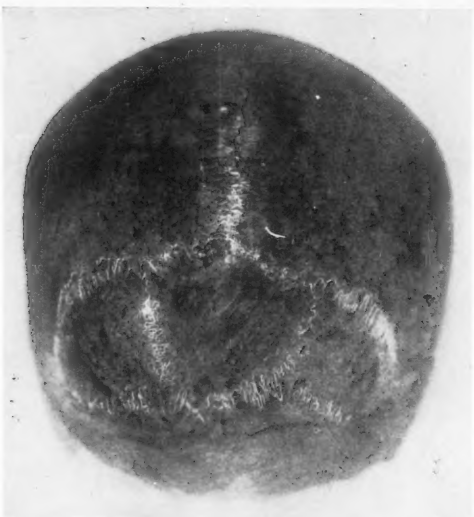


FIG. 32 (No. 103).

Special Burmese Crania, illustrating abnormalities.

Fig. 30. R. *os triangulare* of interparietal and ossicle of lambda.
Fig. 31. Long epipteric bone on right. Fig. 32. Complete tri-partite interparietal.

(i)

(ii)

(iii)

(i) Peruvian Crania, Nos. 999, 1000, Royal College of Surgeons. (ii) New Hebridean Crania, Nos. 1161^{11}, 1161^{7}, Royal College of Surgeons. (iii) Moriori Crania, Nos. 764, 765^{3}, Royal College of Surgeons.
(i) and (ii) Artificially deformed; (iii) Random specimens of a race asserted by Prof. Ruggeri to deform their skulls.

III. 1 with IV. 10.

III. 1 with IV. 10.

IV. 10.

IV. 5.

FIG. 41. Fibula, Tibia and Femur of Beaver (*Castor fiber*).

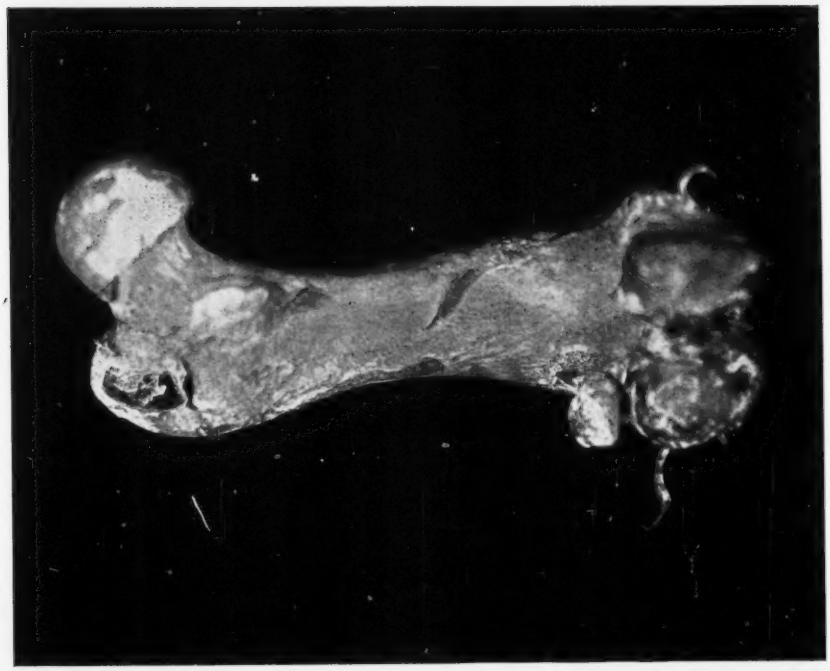


FIG. 40. Femur of Sea Otter (*Enhydra lutris*).

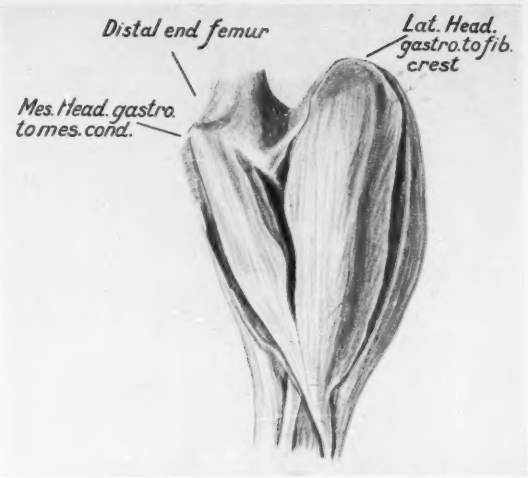


FIG. 42. *Ornithorhynchus* (Musculature).

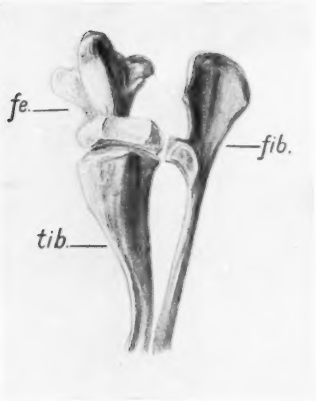


FIG. 43 *a*.

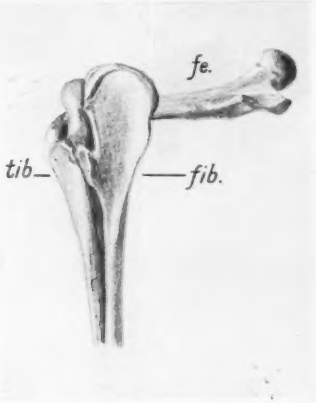


FIG. 43 *b*.

Ornithorhynchus (Bones of Lower Limb).

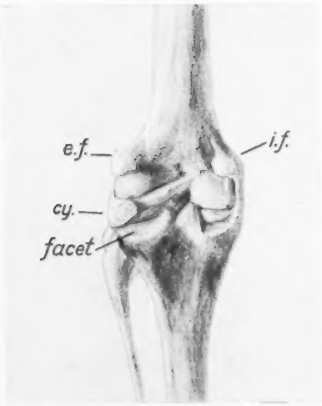


FIG. 44. Cat (*Felis domestica*).

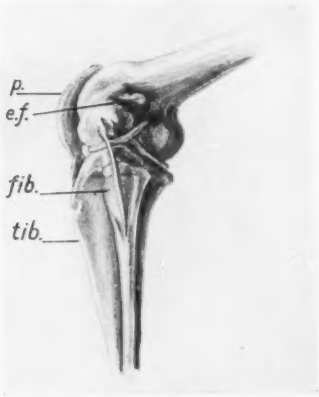


FIG. 45. *Hyrax*.

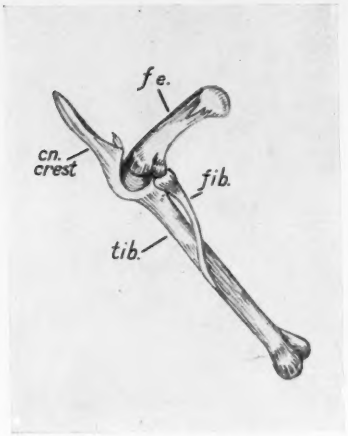


FIG. 46. *Colymbus glacialis* (after Owen).

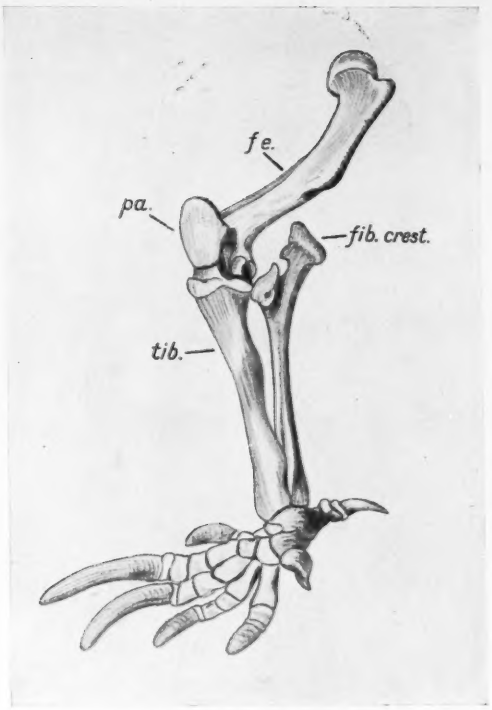


FIG. 47. *Echidna* (after Cuvier).

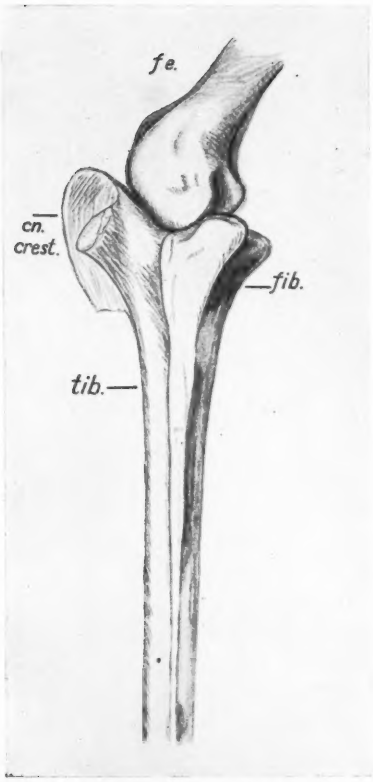


FIG. 48. Cereopsis Goose.

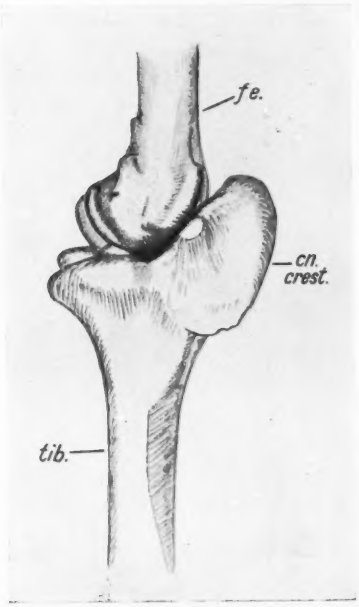


FIG. 49. Cereopsis Goose.

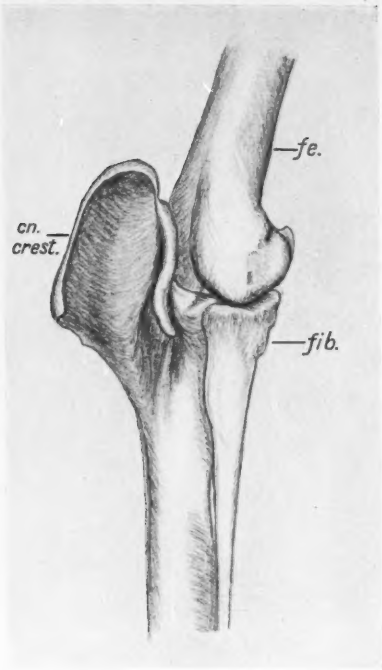


FIG. 50 Albatross (*Diomedea exulans*).

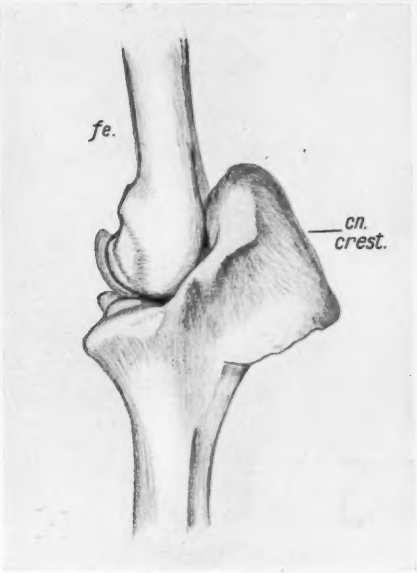


FIG. 51

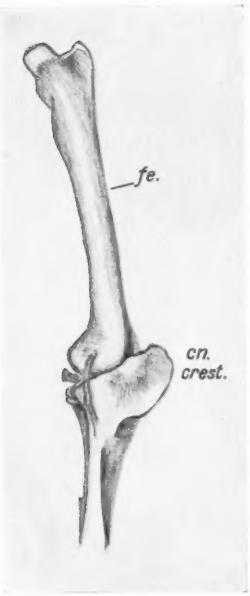


FIG. 52

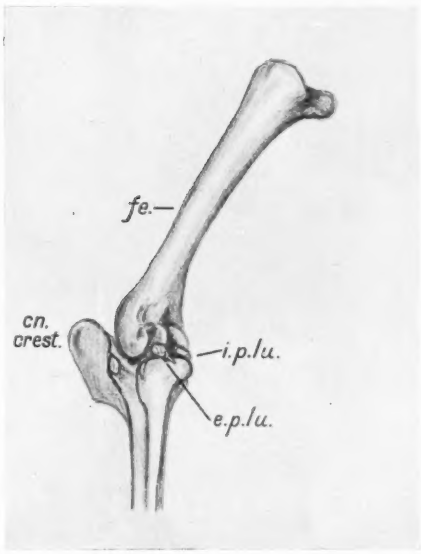


FIG. 53
Dendrocygna.

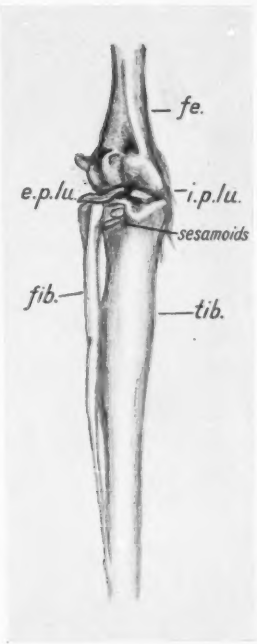


FIG. 54

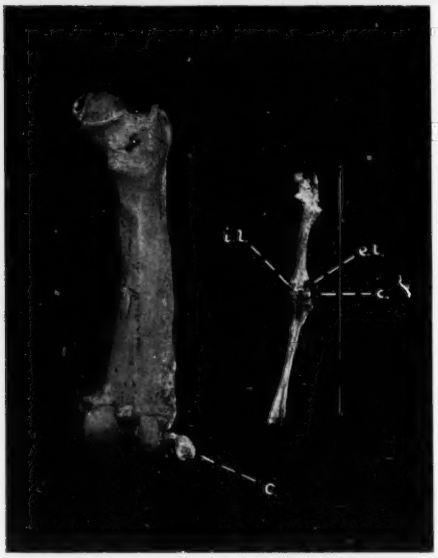


FIG. 55 *a*.
Tamandua Anteater
(*Tamandua tetradactyla*).

FIG. 55 *b*.
Common Squirrel
(*Sciurus vulgaris*).

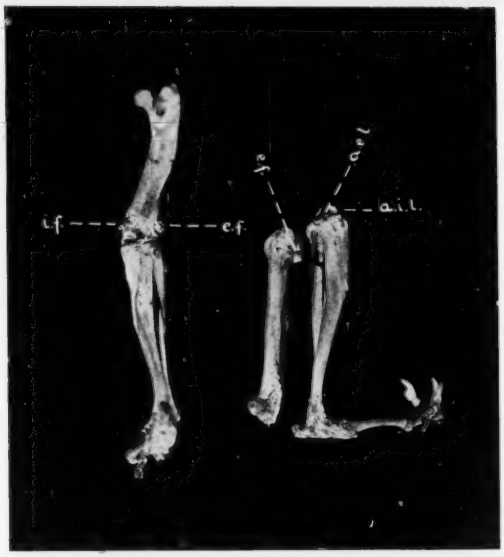


FIG. 56 *a*.

FIG. 56 *b*.
Common Hedgehog (*Erinaceus europaeus*)
in two aspects.

FIG. 57 *a*. Golden-bellied Rat
(*Hydromys chrysogaster*).

FIG. 57 *b*. Musquash
(*Fiber zibethicus*).

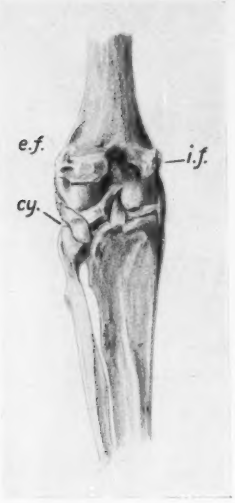


FIG. 58. Common rabbit (*Lepus cuniculus*).

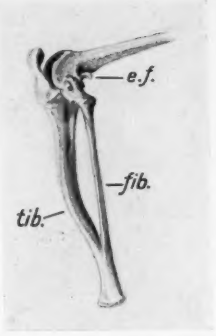


FIG. 59. *Potamogale velox* (West Africa).

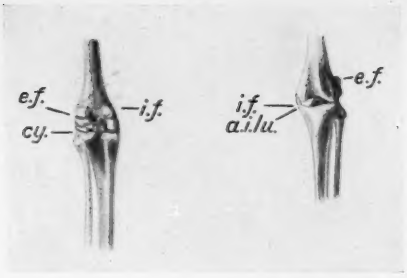


FIG. 60 *a*. FIG. 60 *b*.
Paraxerus jacksoni (African Squirrel).

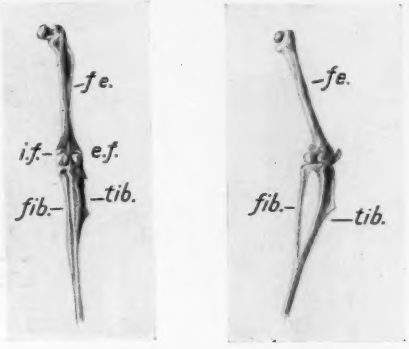


FIG. 62 *a*. FIG. 62 *b*.
Hylomys suillus dorsalis (Borneo).

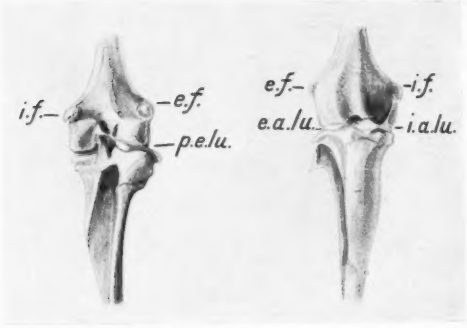


FIG. 61 *a*. FIG. 61 *b*.
Musquash (*Fiber zibethicus*).

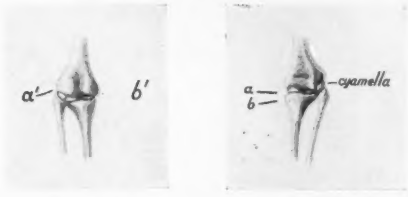
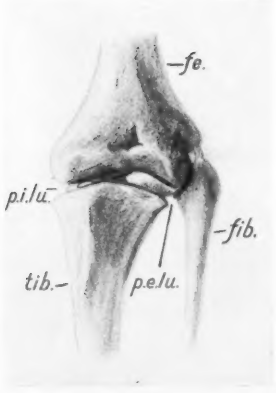


FIG. 63 *a*. FIG. 63 *b*.
Trachysaurus rugosus (Australian Lizard).



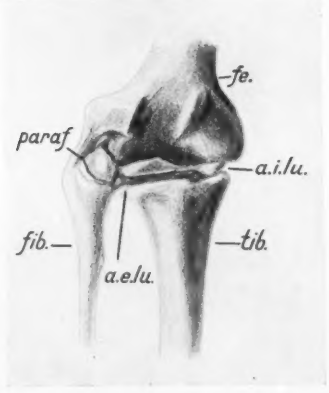


FIG. 64 *a*. FIG. 64 *b*.
Nile Monitor (*Varanus niloticus*).

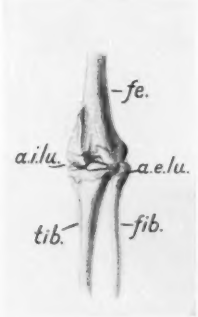


FIG. 65 *a*. FIG. 65 *b*.
Amphibolurus barbata.

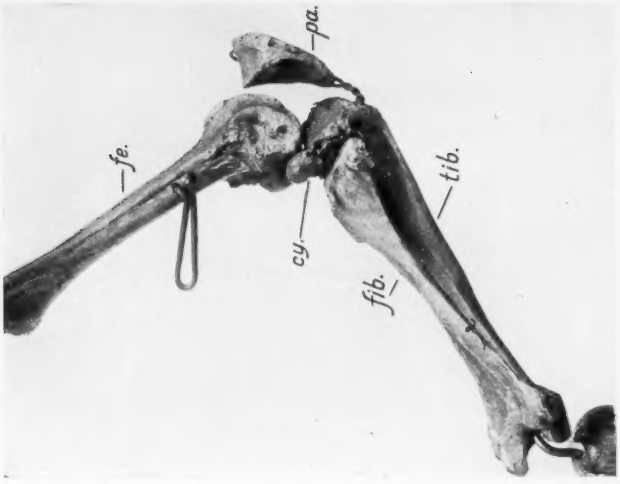


FIG. 67. Chinese Pangolin (*Manis aurita*).

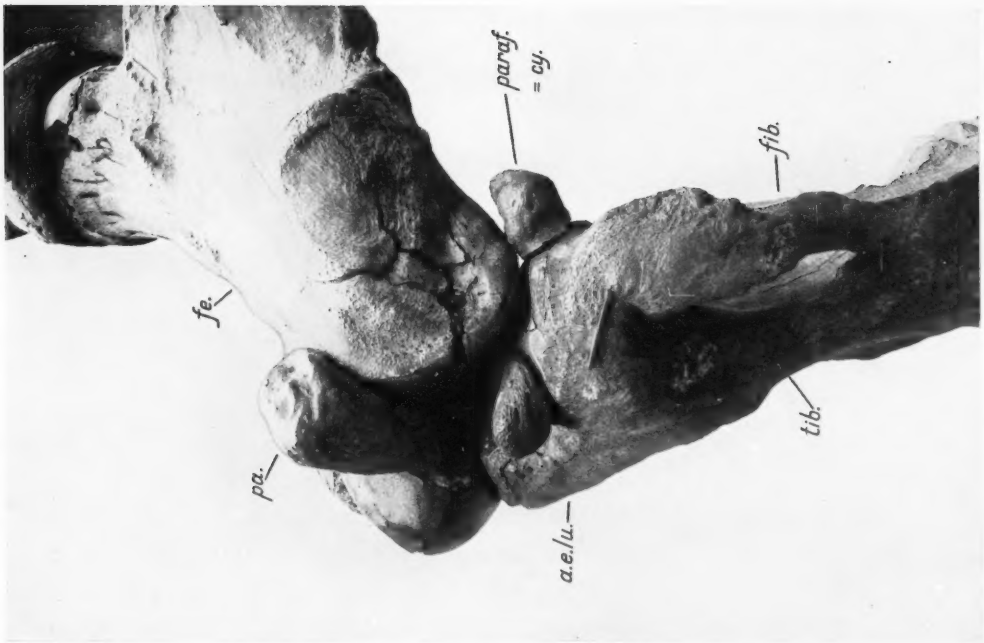


FIG. 66. Fossil Giant Sloth in R. C. of S.'s Museum (*Megatherium cuvieri*).

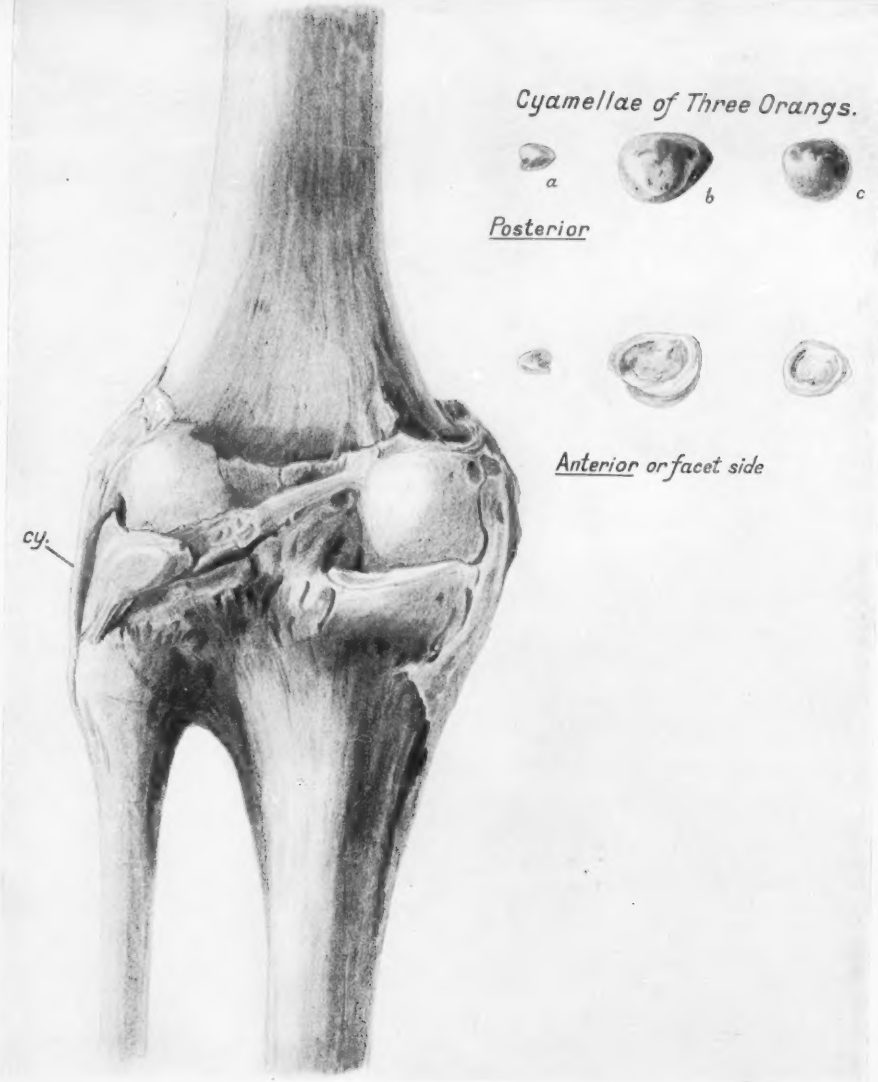


FIG. 68. Orang. Cyamella shown in position.

FIG. 69. Orang. Cyamella raised from its facet.

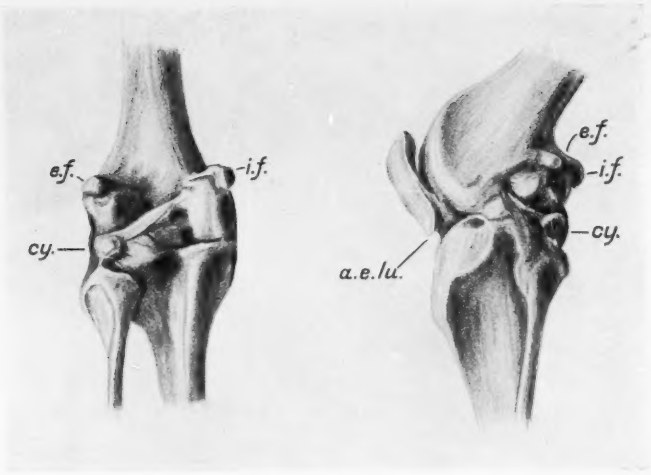


FIG. 70 *a*. FIG. 70 *b*. *Lemur varia* in two aspects.

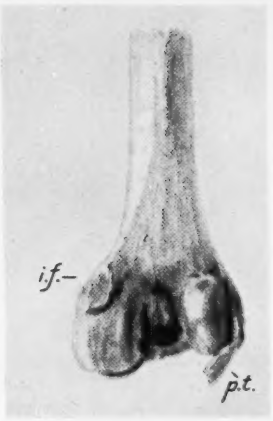


FIG. 72. Gibbon.

FIG. 71 *a*. *Mycetis palliatus*.

FIG. 73. *Lemur melanocephalus*.

FIG. 71 *b*. *Mycetis palliatus*.

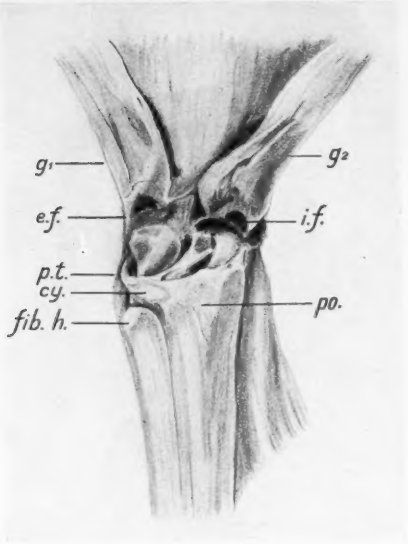


FIG. 74 *a*. Aye-Aye (*Chiromys*).

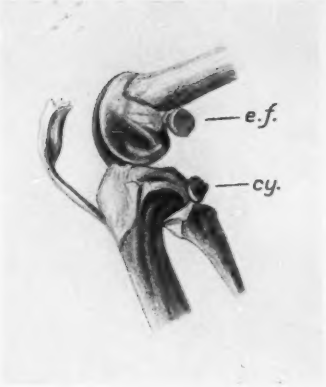


FIG. 74 *b*. *Chiromys* (after Owen).

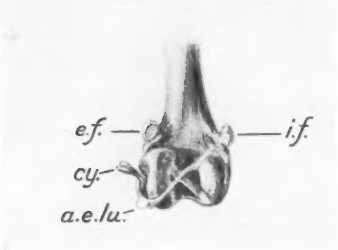


FIG. 75. *Lemur catta*.

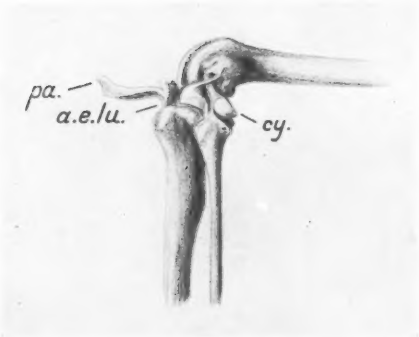


FIG. 76. *Perodicticus potto*.

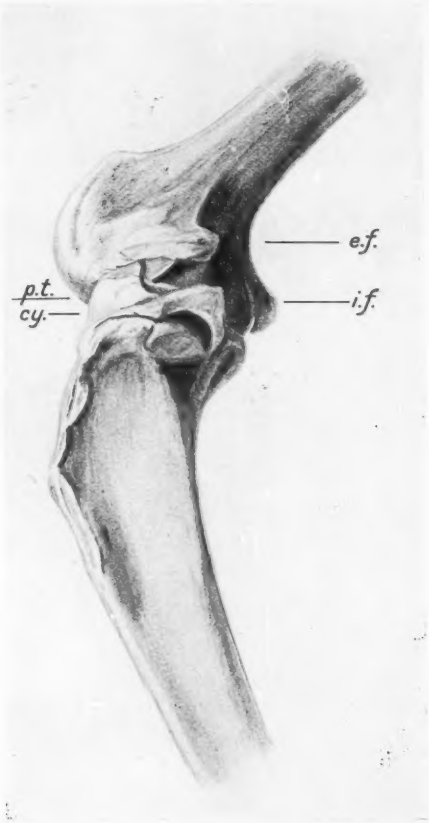


FIG. 77. *Indris brevicaudata*.

FIG. 78

Tasmanian Devil (*Sarcophilus ursinus*).
Parafibula fused.

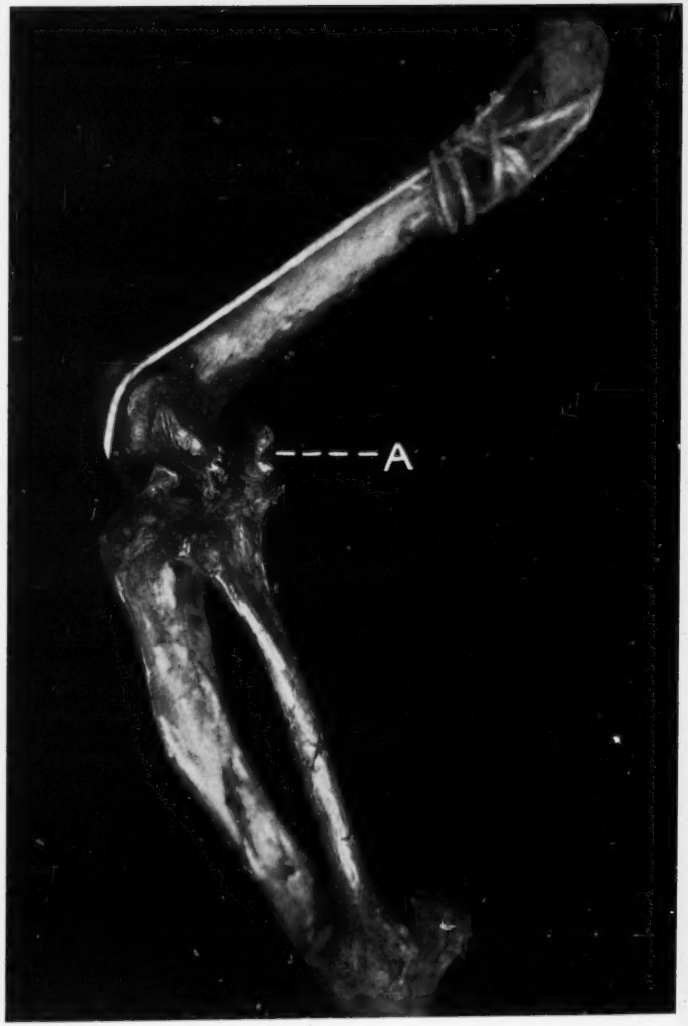


FIG. 79. Wombat. Parafibula free.

Wombat (*Phascolomys wombat*).
Parafibula fused.

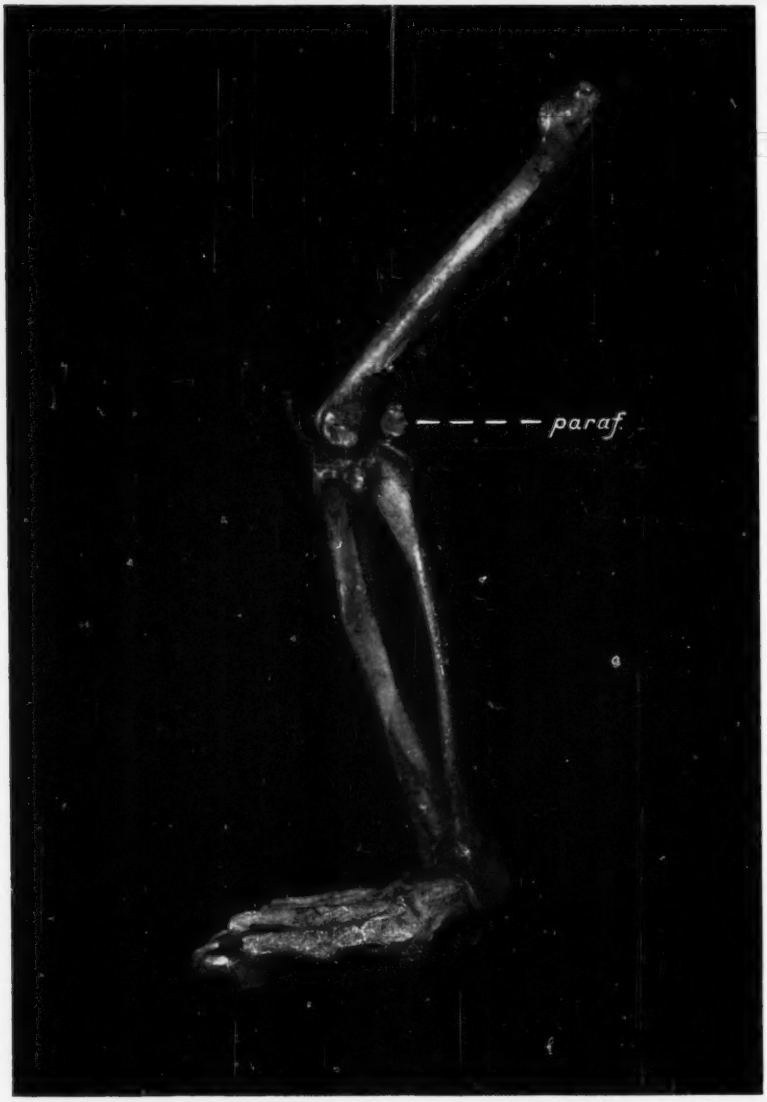


FIG. 80. *Phalangista vulpina.* Free parafibula.

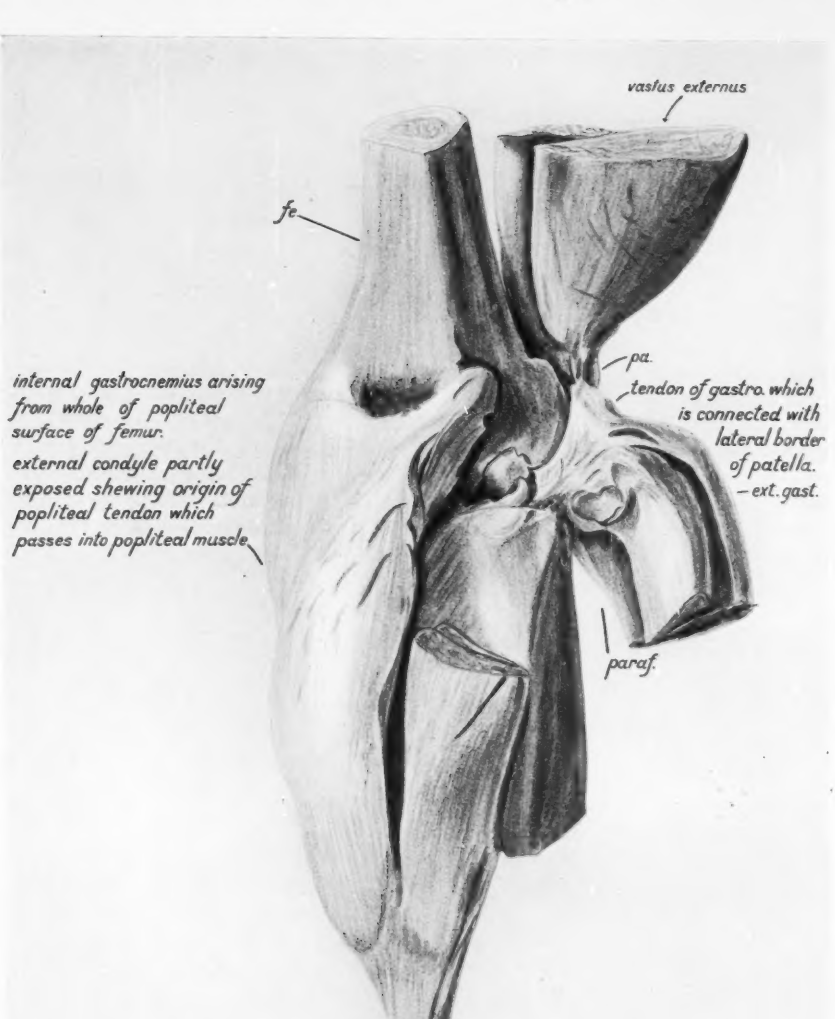

vastus externus
fe
pa.
tendon of gastro. which is connected with lateral border of patella.
– ext. gast.
internal gastrocnemius arising from whole of popliteal surface of femur.
external condyle partly exposed shewing origin of popliteal tendon which passes into popliteal muscle
paraf.
–Tendo Achillis

FIG. 81. Knee-joint of *Phascolomys* (Wombat) with attached muscles.

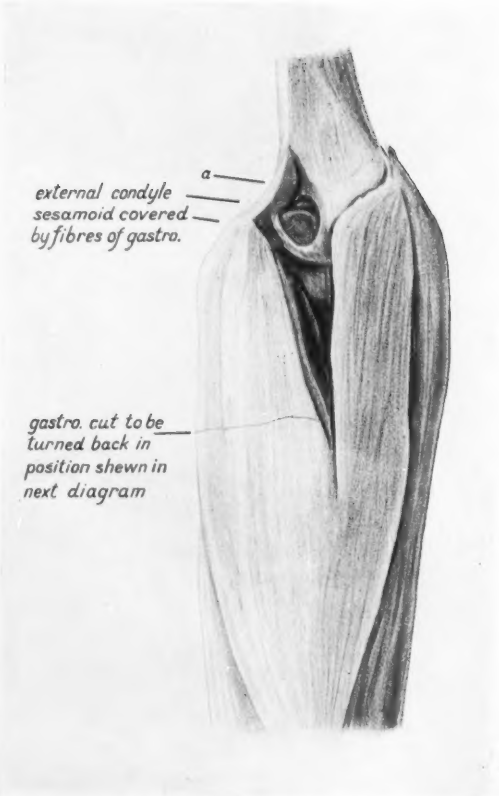


FIG. 82 *a*.

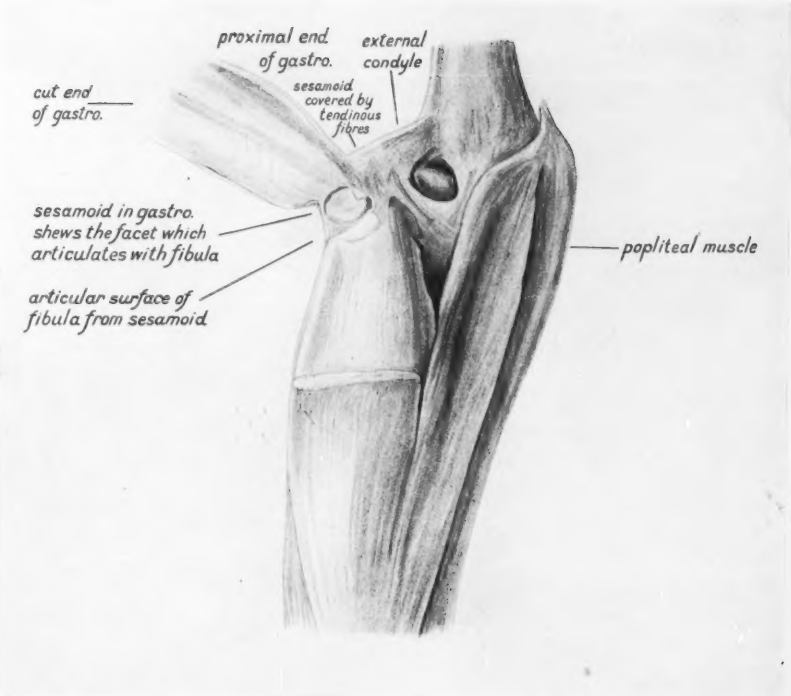


FIG. 82 *b*.

Koala (*Phascolarctus cinereus*). Sesamoid articulating with fibula.

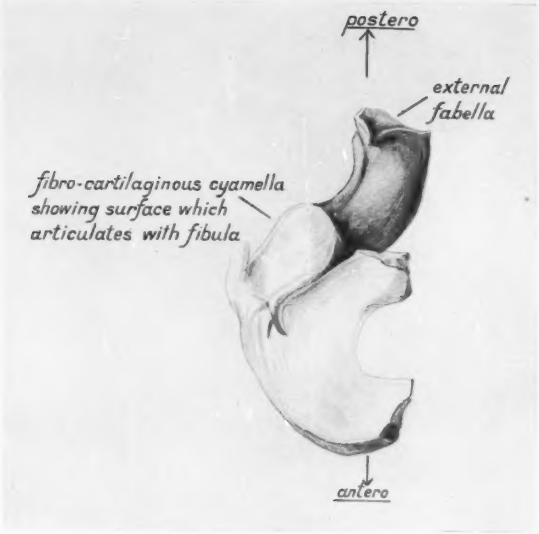


FIG. 83 *a*. *Macropus* sp.?

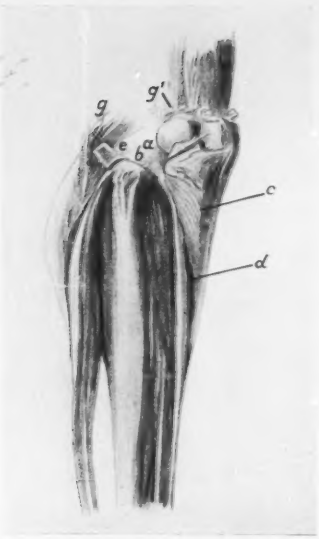


FIG. 84. Virginian Opossum (*Didelphys virginiana*).

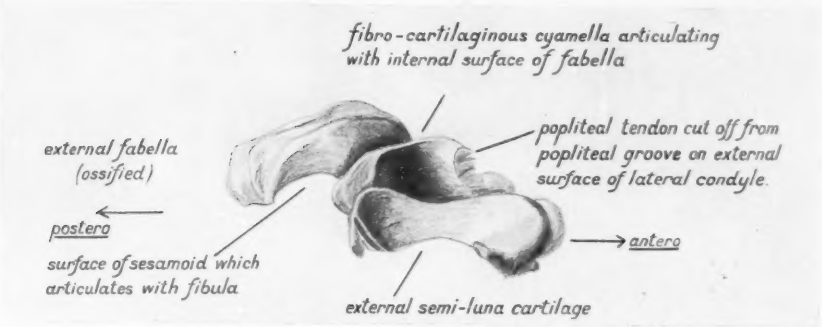

fibro-cartilaginous cyamella articulating with internal surface of fabella
external fabella (ossified)
popliteal tendon cut off from popliteal groove on external surface of lateral condyle
postero
antero
surface of sesamoid which articulates with fibula
external semi-luna cartilage

FIG. 83 *b*. *Macropus* sp.?

FIG. 85. *Phalangista vulpina*.

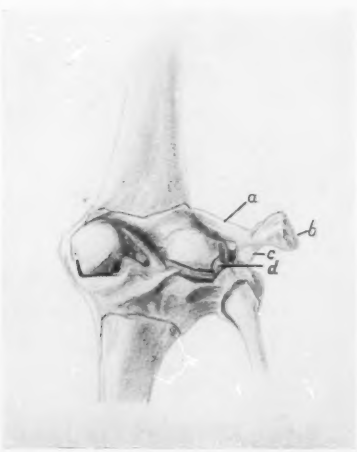


FIG. 86. Koala (*Phascolarctus cinereus*).

FIG. 87. Wallaby (*Macropus bennettii*).

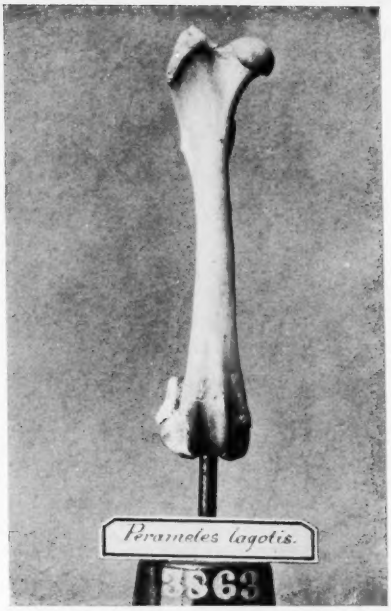


FIG. 88. Rabbit-eared Perameles.

Doubtful Position of Parafibula.

FIG. 89 *a*. Tasmanian Devil (*Sarcophilus ursinus*). Anterior Aspect.

FIG. 89 *b*. Tasmanian Devil (*Sarcophilus ursinus*). Lateral Aspect.

Correct Position of Parafibula.

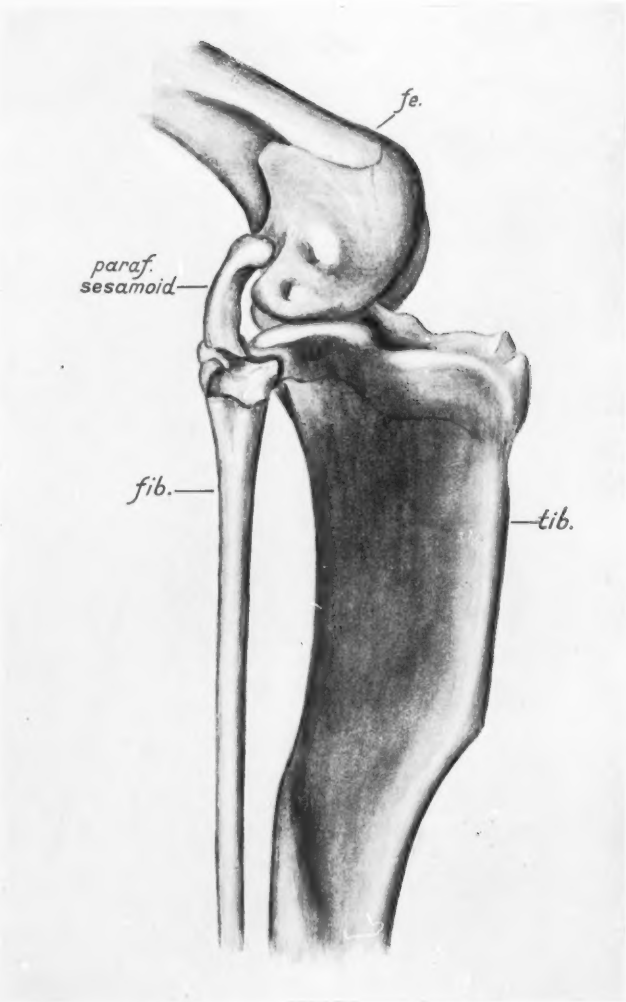


FIG. 90. Wallaby (*Halmaturus* sp.?).

FIG. 91. *Corucia zebrata*. *a*, parafibular sesamoid; *b*, paratibial sesamoid united by ligament.

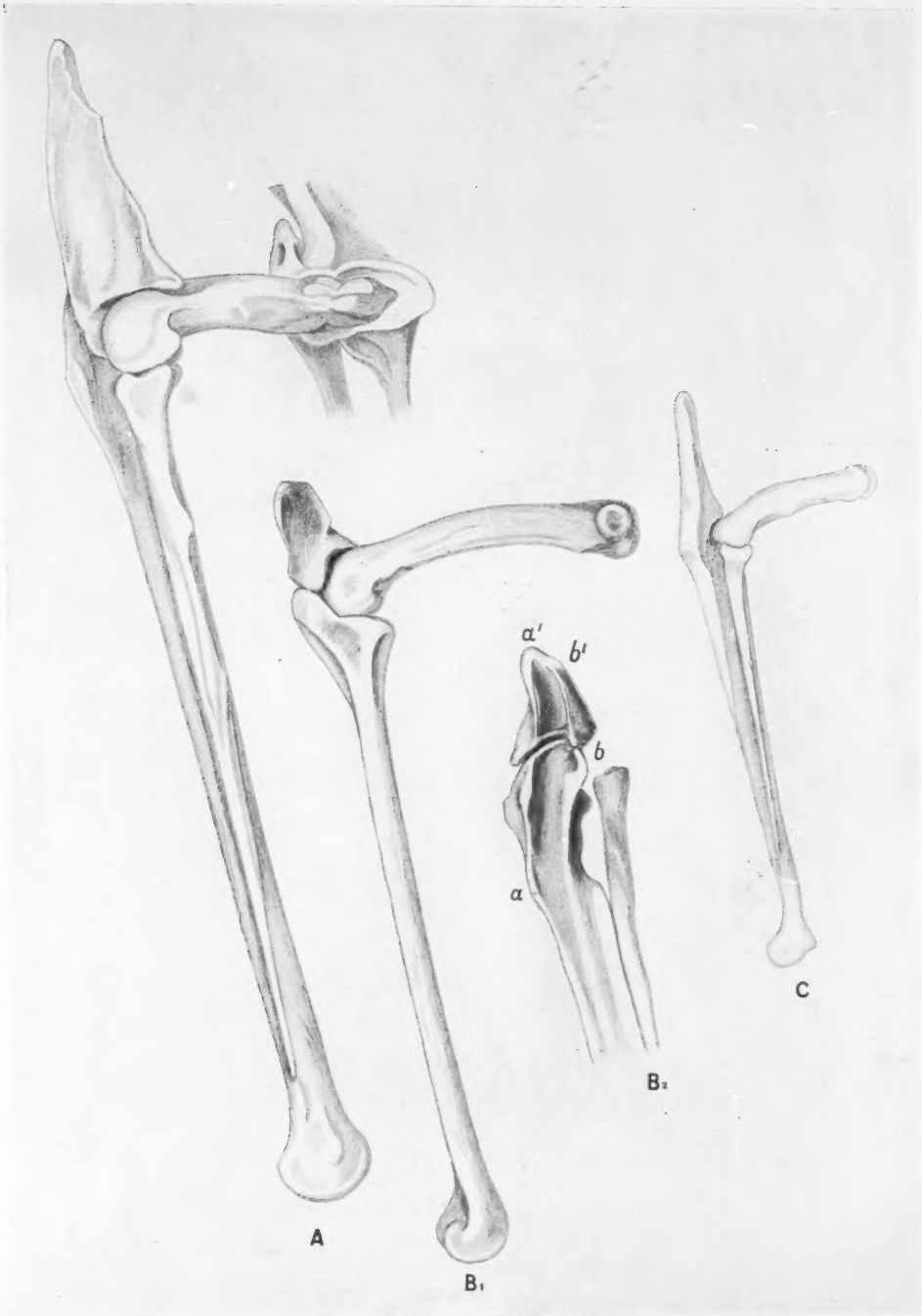


FIG. 92. FIG. 93 *a*. FIG. 93 *b*. FIG. 94.

A. *Hesperornis regalis*, after Marsh ($\frac{1}{4}$ natural size). B_1. R. Limb, mesial aspect from actual limb of Cormorant (natural size). B_2. L. Limb, lateral aspect from actual limb of Cormorant (natural size).
C. *Colymbus torquatus*, after Coues ($\frac{1}{2}$ natural size).

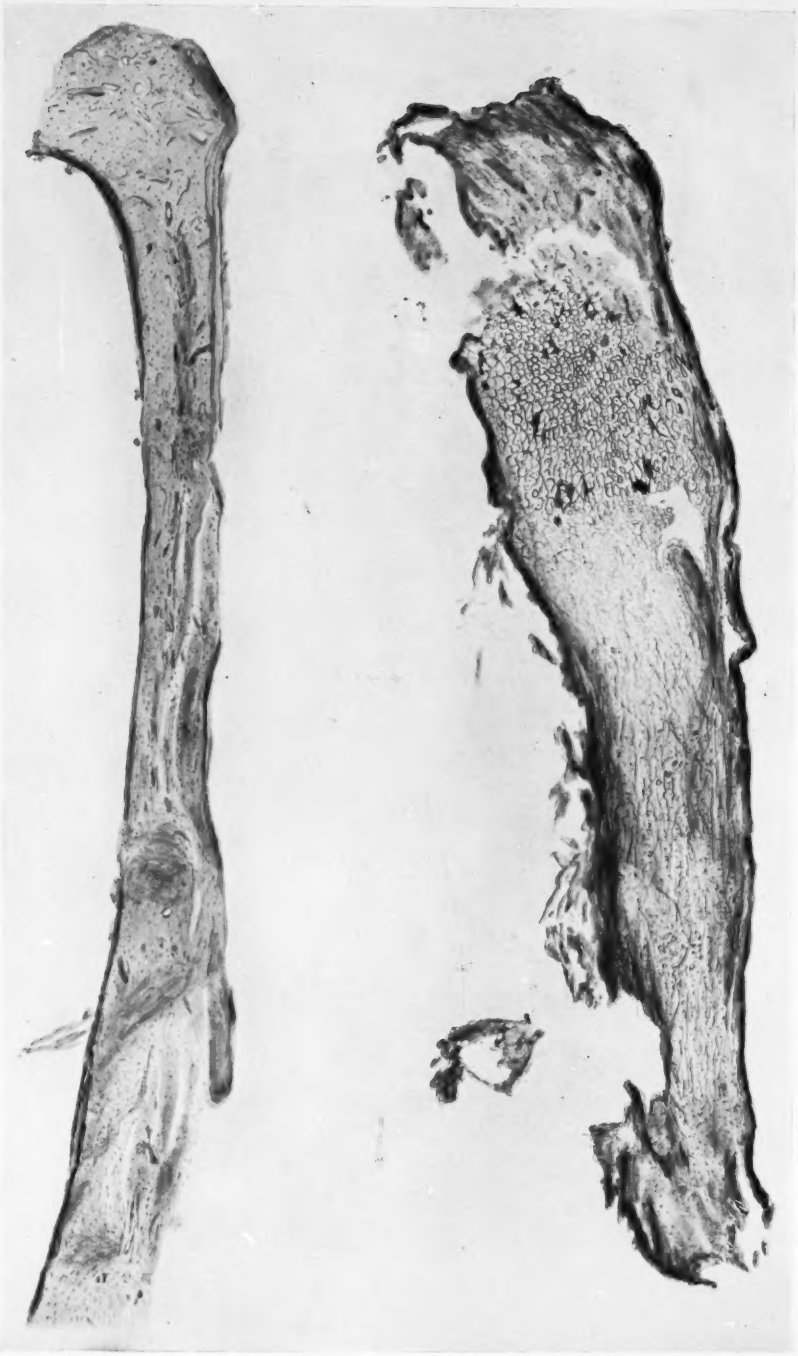


FIG. 95. Section of Fibular Crest of Young Platypus. × c. 60.

FIG. 96. Section of Fibular Crest of Embryo Platypus. × c. 60.

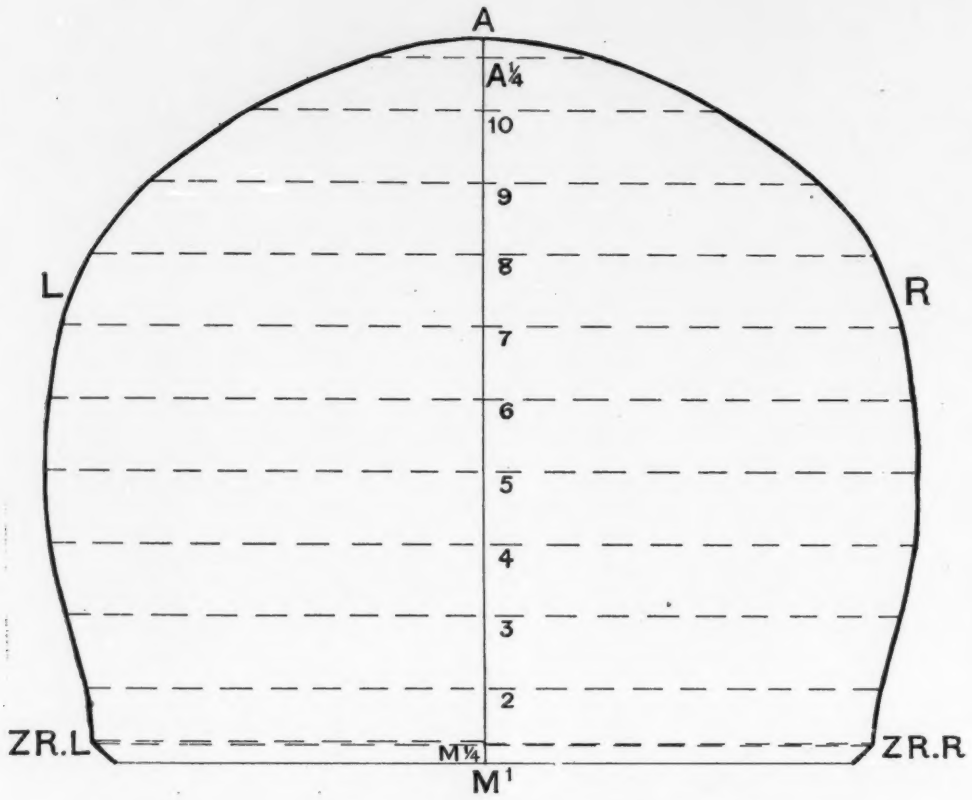


FIG. I. Burmese Crania. Type A (Burmans). ♂. Vertical Contour. (44 crania.)

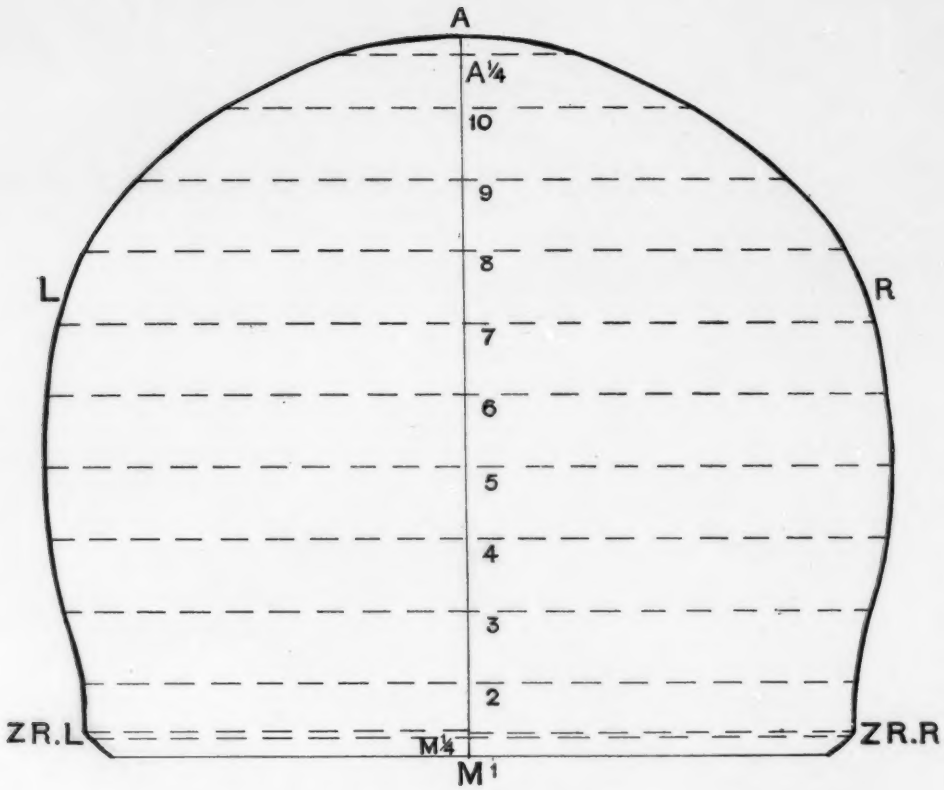


FIG. II. Burmese Crania. Type B (? Hybrids). ♂. Vertical Contour. (7 crania.)

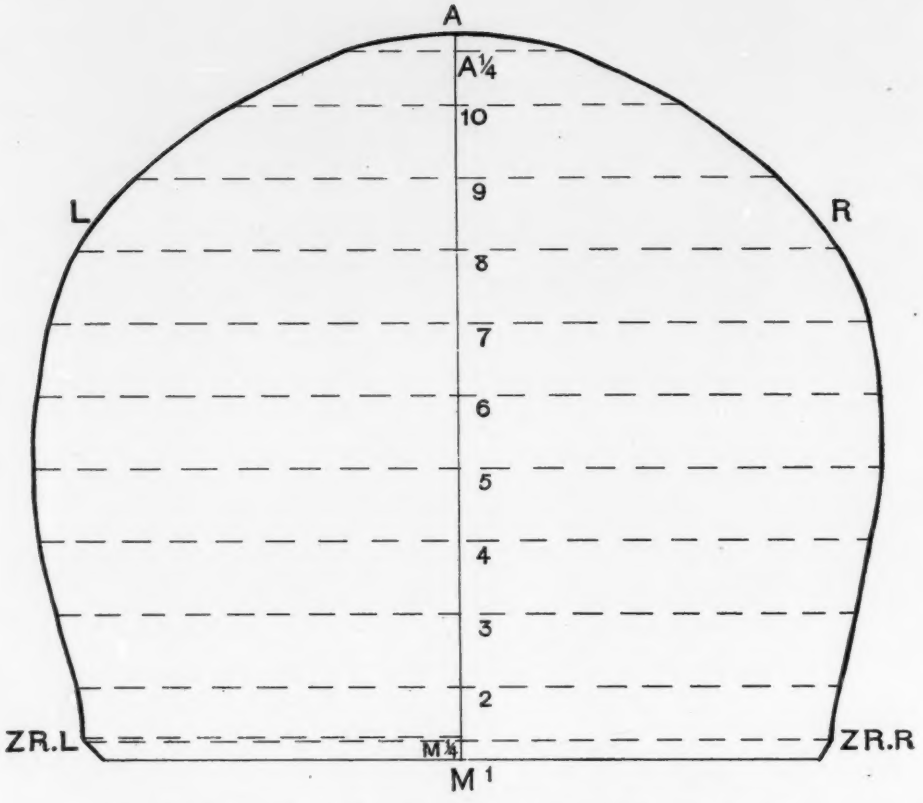


FIG. III. Burmese Crania. Type C (? Karens). ♂. Vertical Contour. (8 crania.)

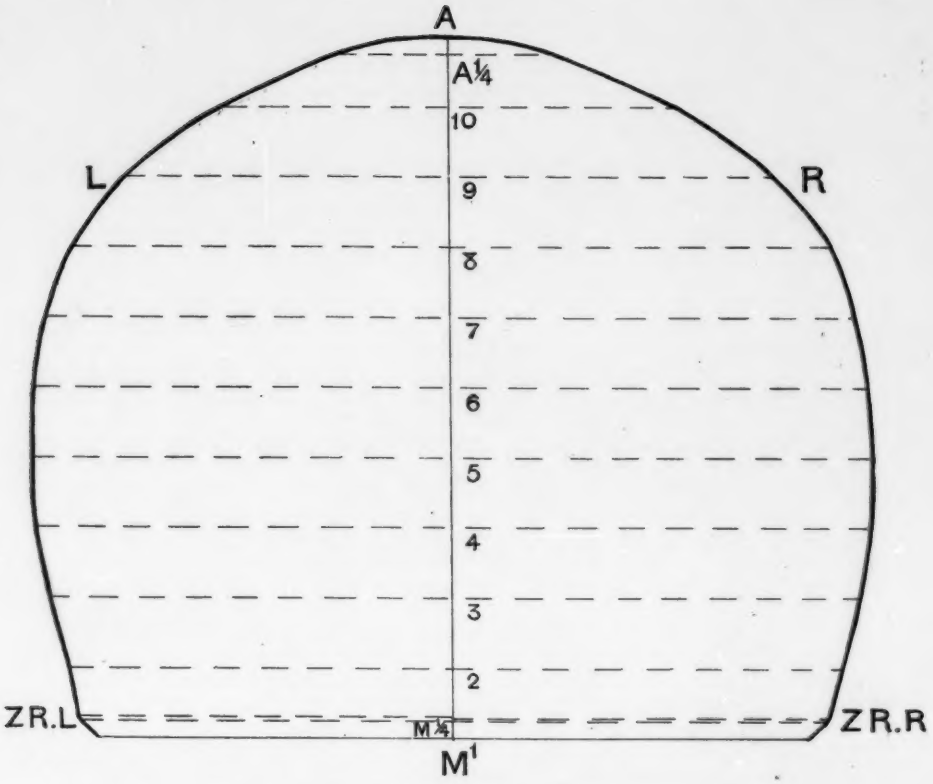


FIG. IV. Burmese Crania. Type A (Burmans). ♀. Vertical Contour. (38 crania.)

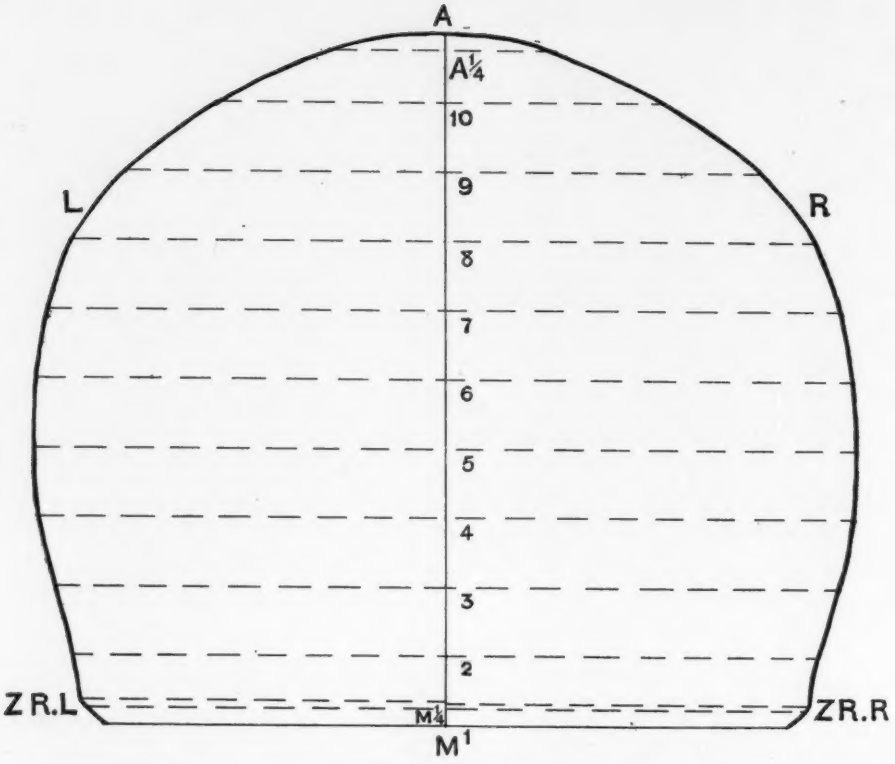


FIG. V. Burmese Crania. Type B (? Hybrids). ♀. Vertical Contour. (17 crania.)

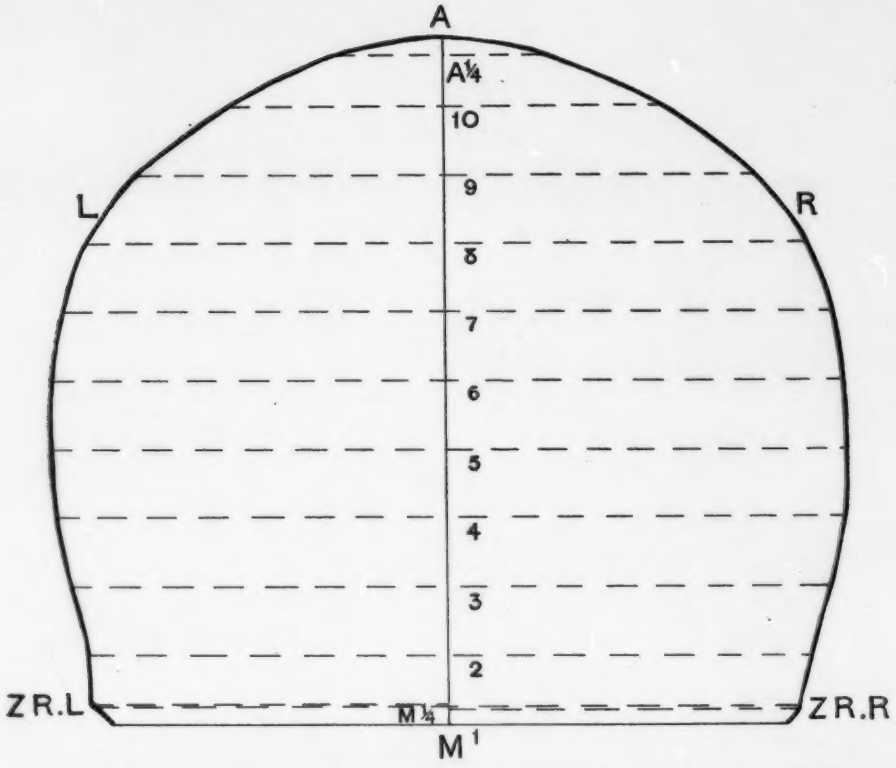


FIG. VI. Burmese Crania. Type C (? Karens). ♀. Vertical Contour. (18 crania.)

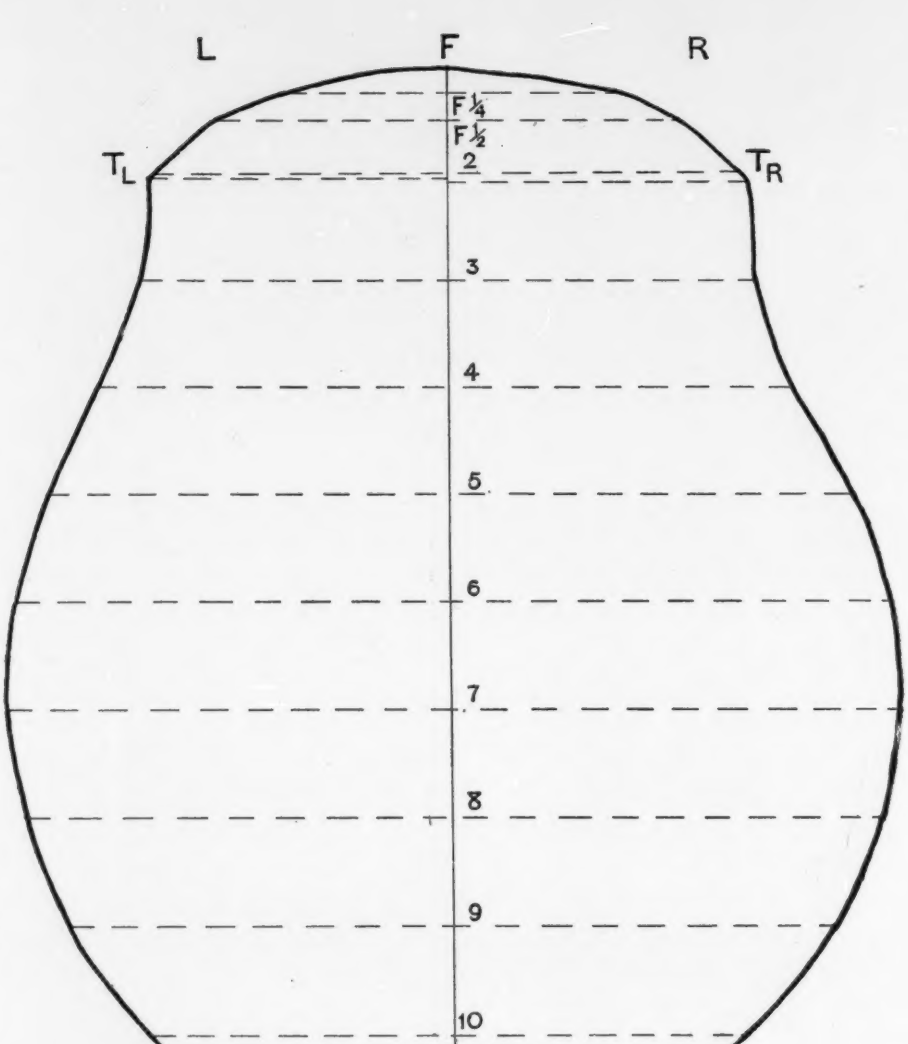


FIG. VII. Burmese Crania. Type A (Burmans). ♂. Horizontal Contour. (44 crania.)

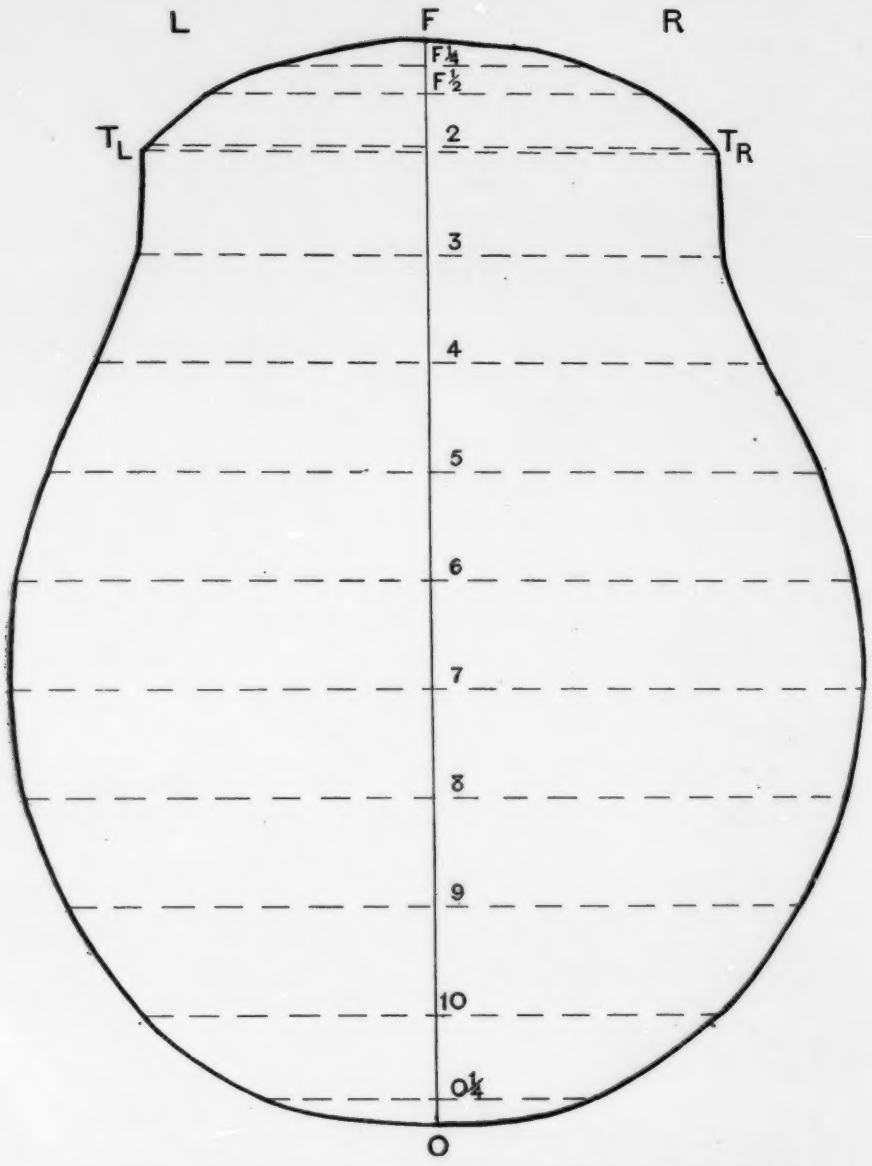


FIG. VIII. Burmese Crania. Type B (? Hybrids). ♂. Horizontal Contour. (7 crania.)

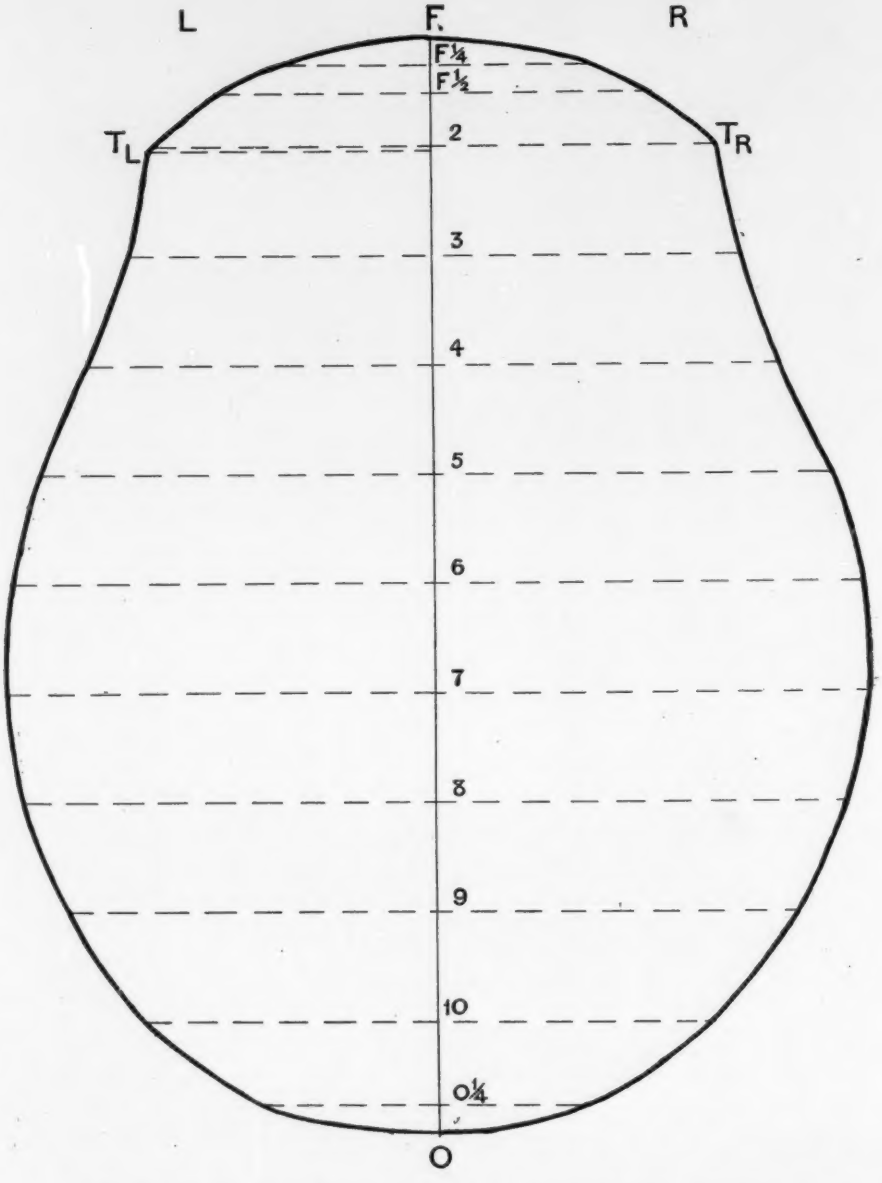


FIG. IX. Burmese Crania. Type C (? Karens). ♂. Horizontal Contour. (8 crania.)

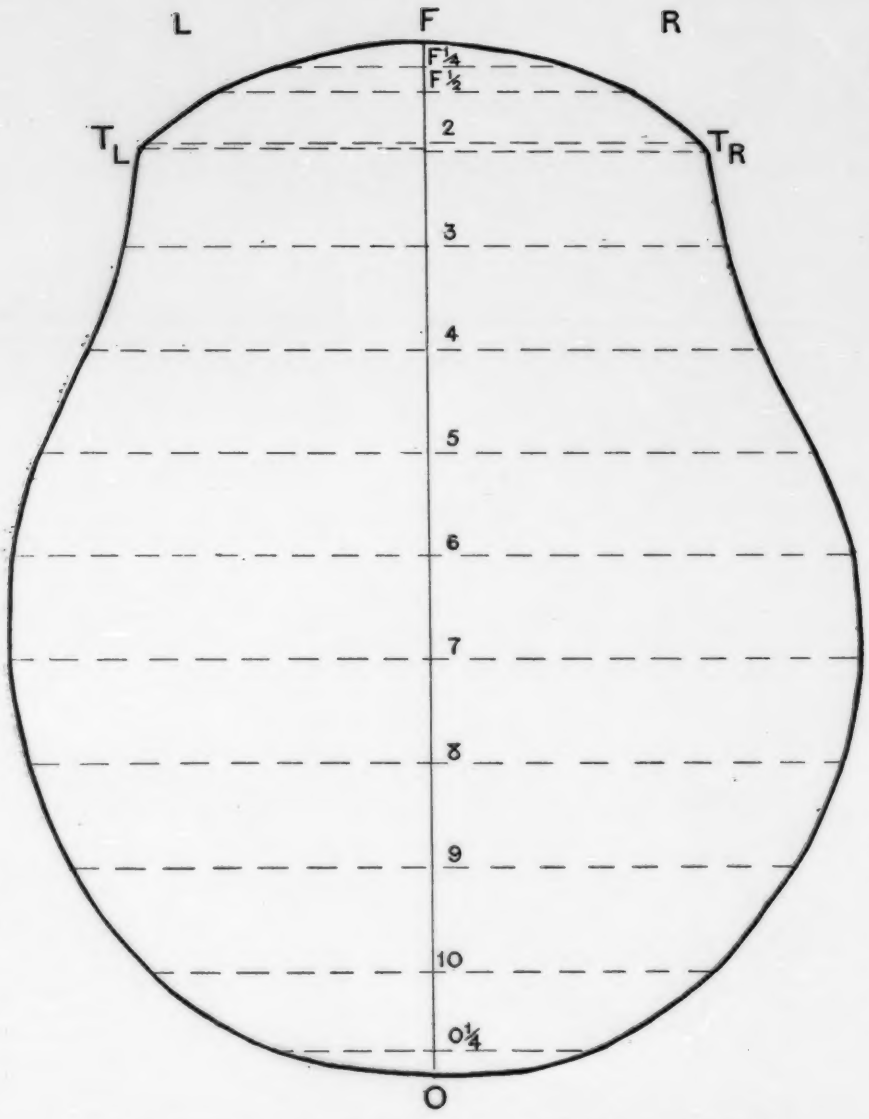


FIG. X. Burmese Crania. Type A (Burmans). ♀. Horizontal Contour. (39 crania.)

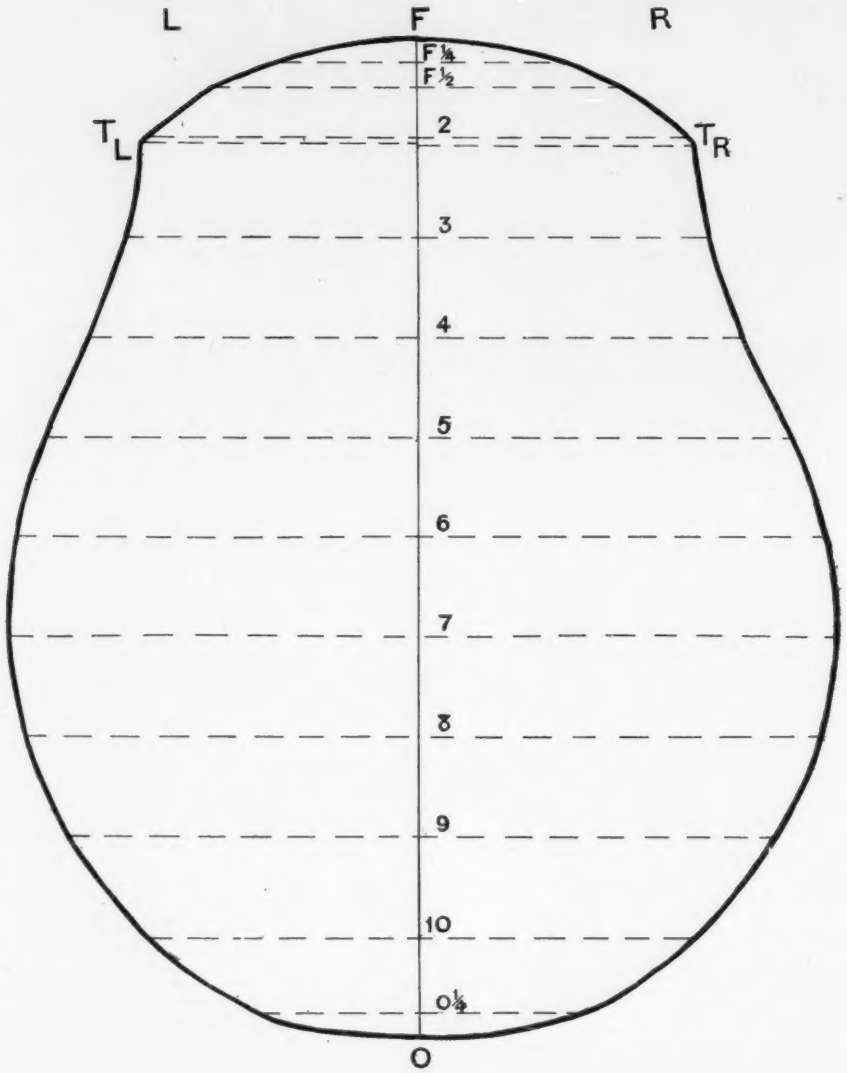


FIG. XI. Burmese Crania. Type B (? Hybrids). ♀. Horizontal Contour. (17 crania.)

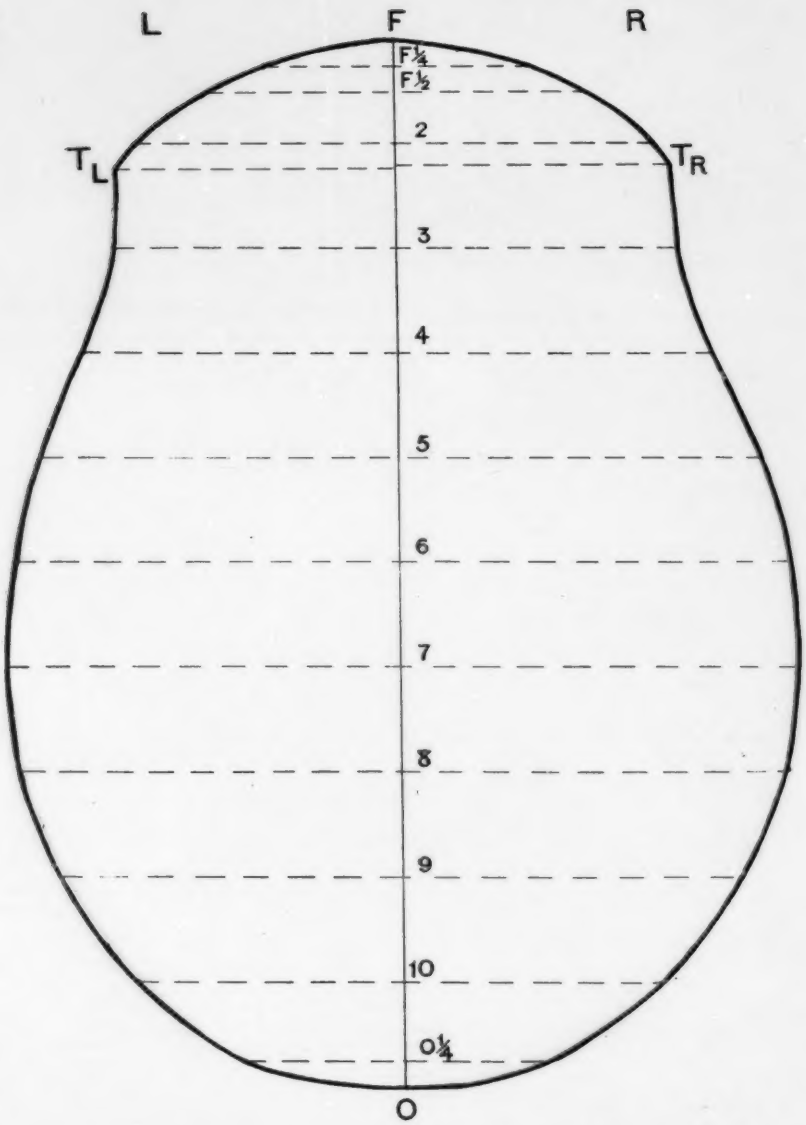


FIG. XII. Burmese Crania. Type C (? Karens). ♀. Horizontal Contour. (18 crania.)

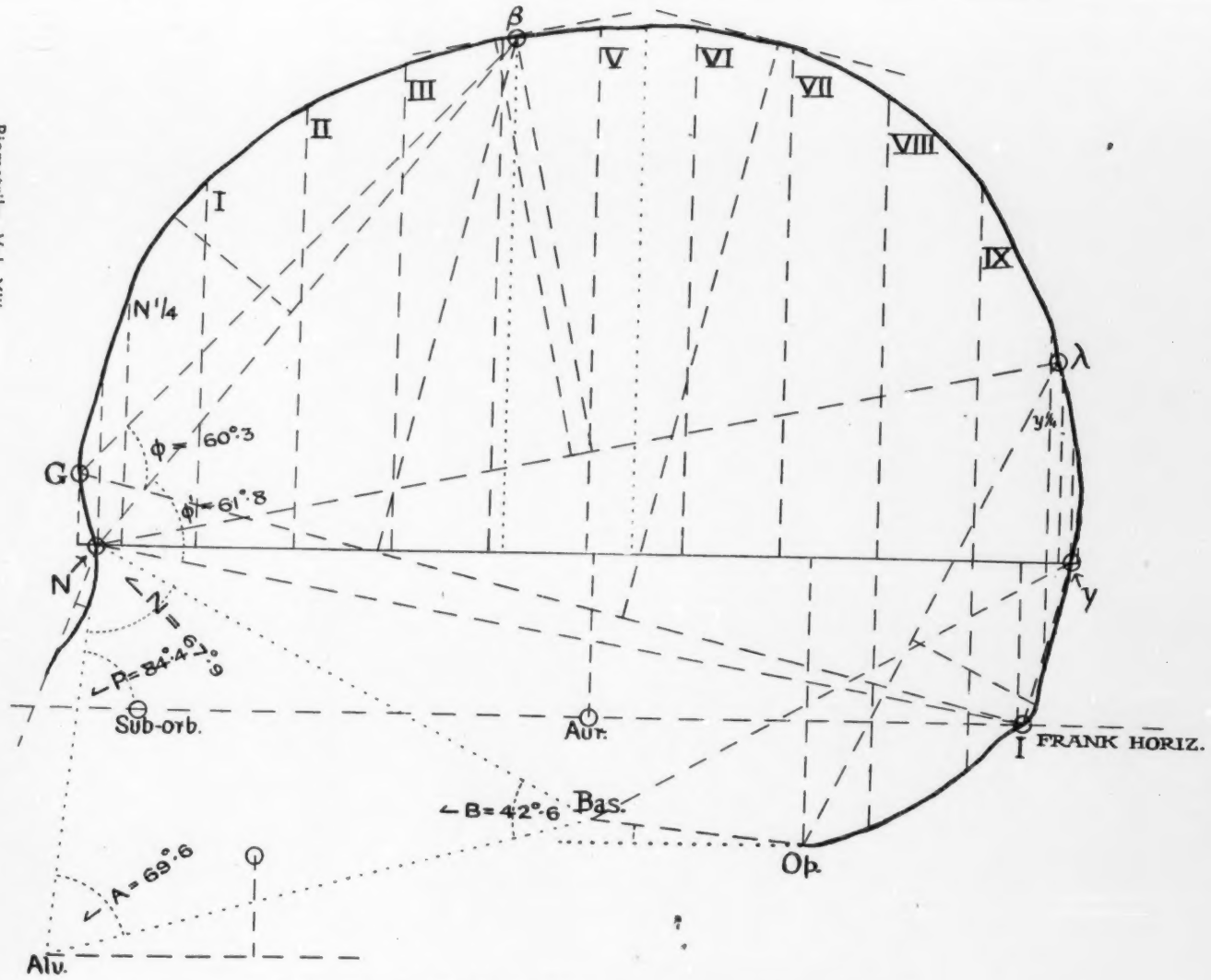


FIG. XIII. Burmese Crania. Type A (Burmans). ♀. Sagittal Contour. (44 crania.)

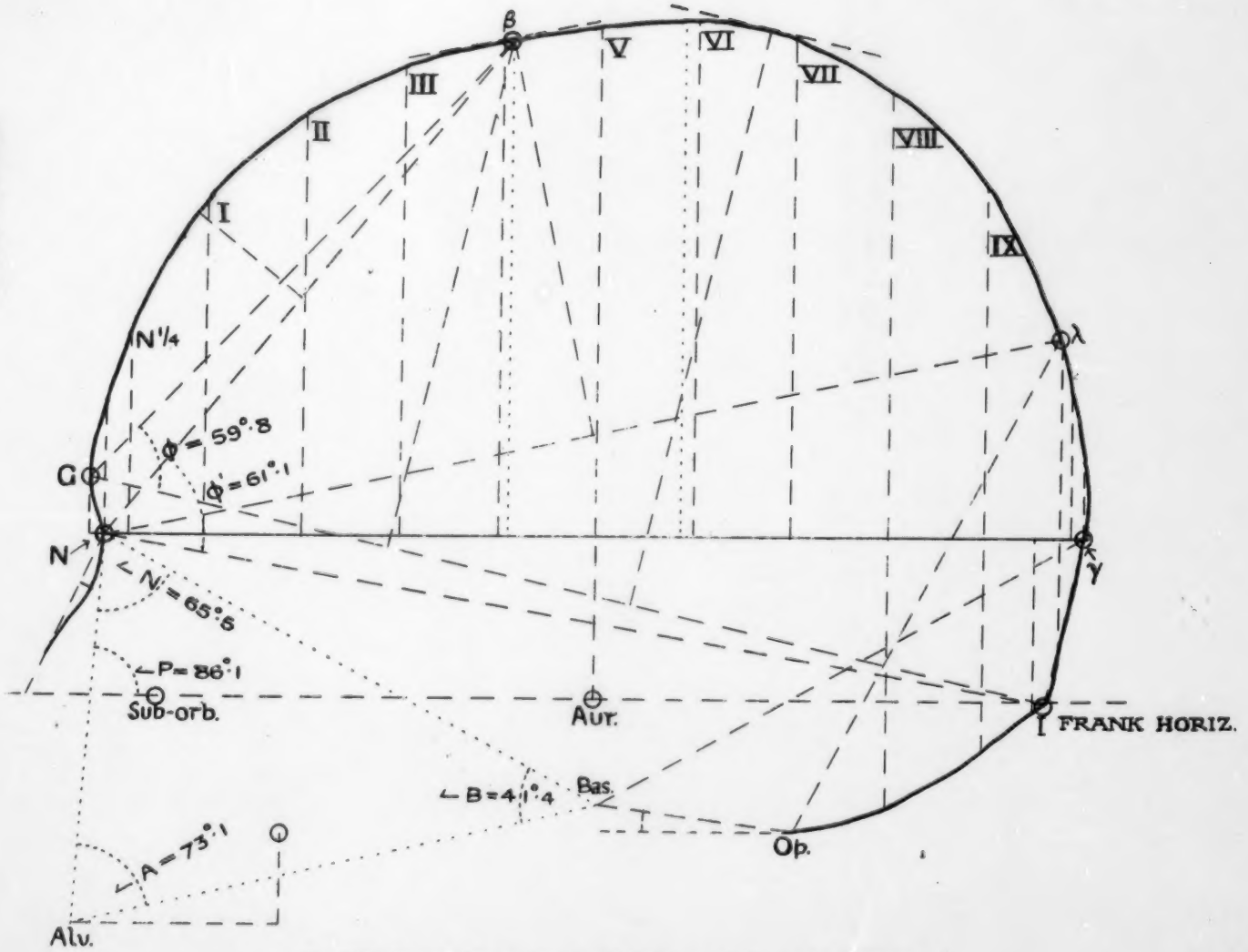


FIG. XIV. Burmese Crania. Type B (? Hybrids). ♂. Sagittal Contour. (7 crania.)

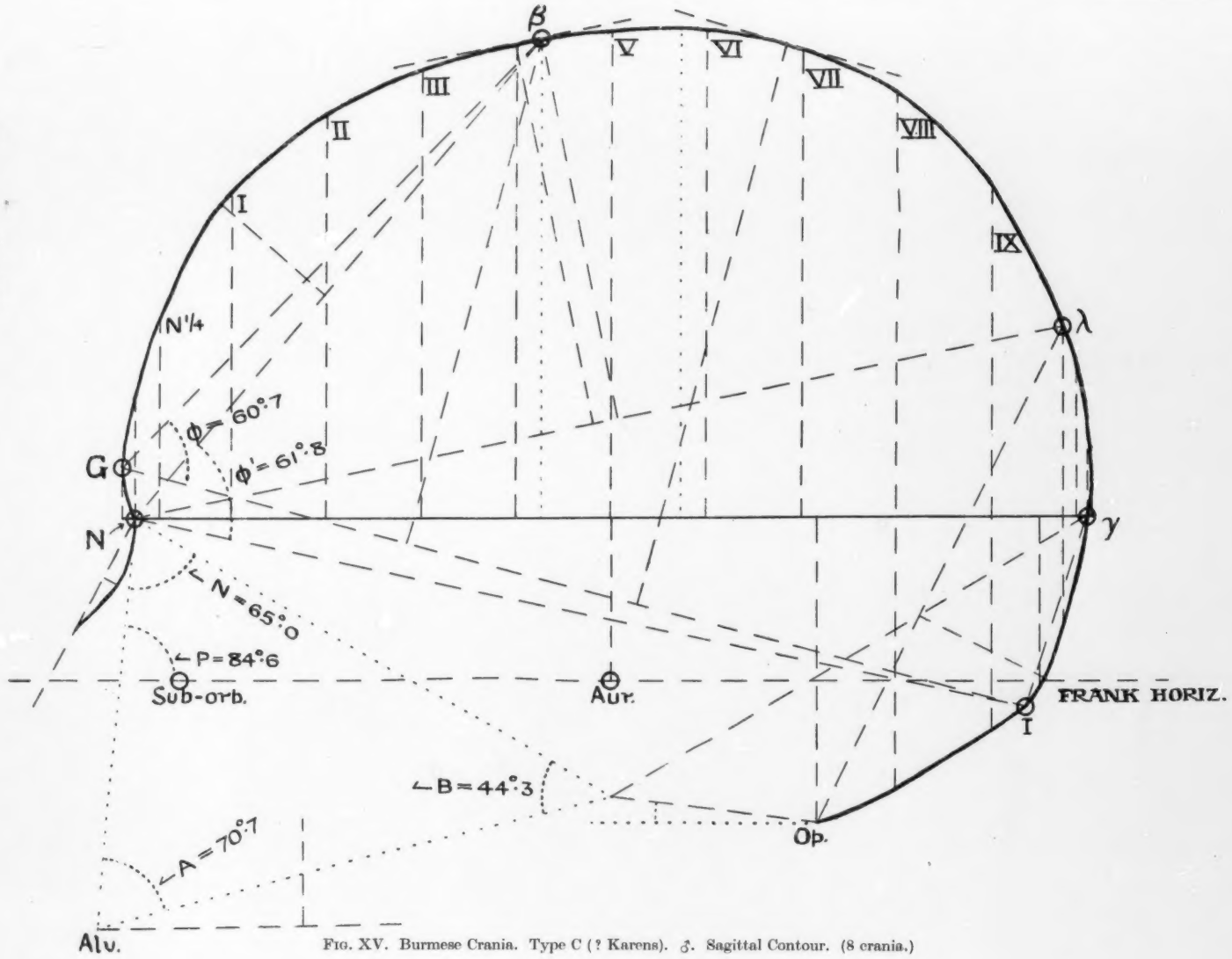


FIG. XV. Burmese Crania. Type C (? Karens). ♂. Sagittal Contour. (8 crania.)

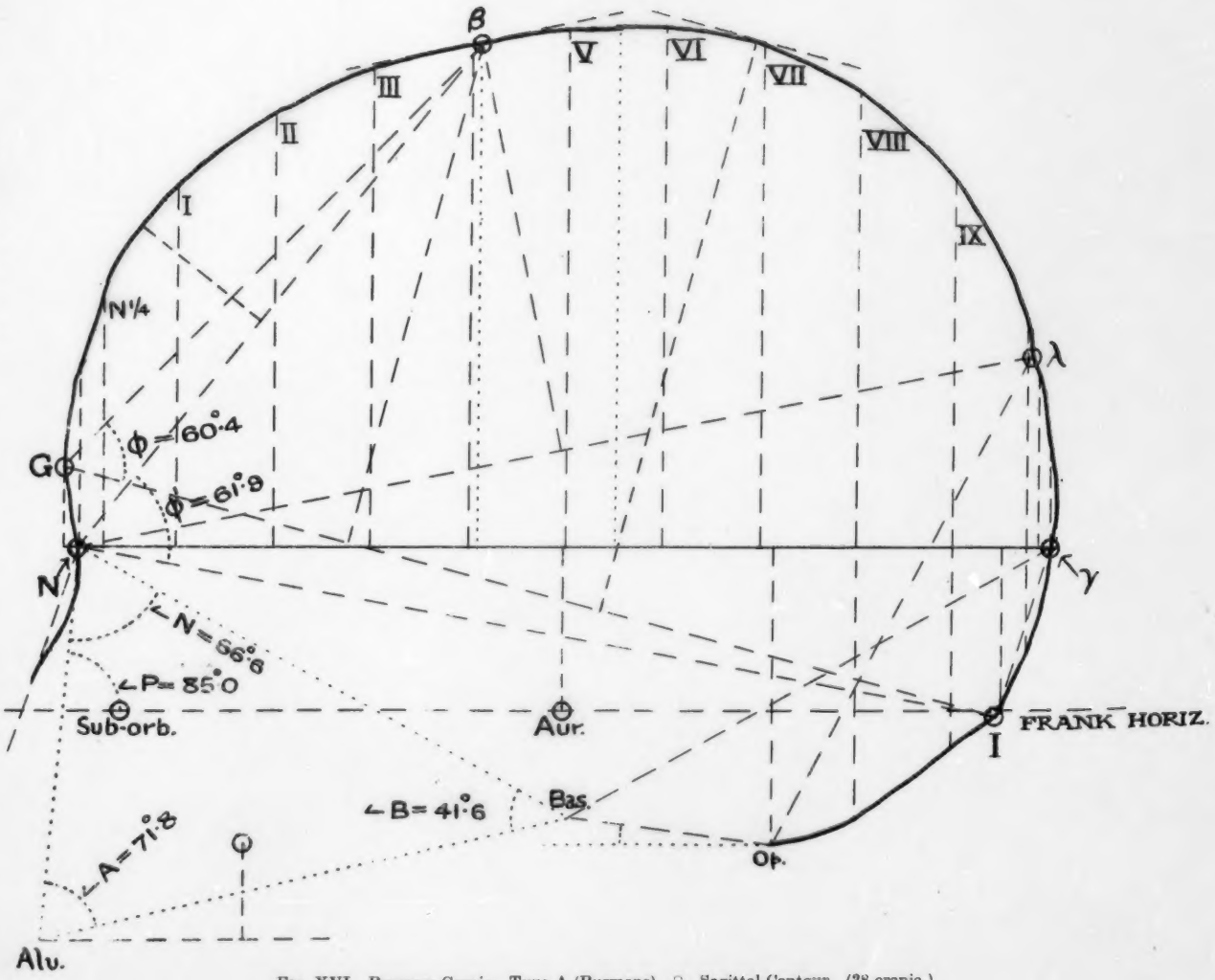


FIG. XVI. Burmese Crania. Type A (Burmans). ♀. Sagittal Contour. (38 crania.)

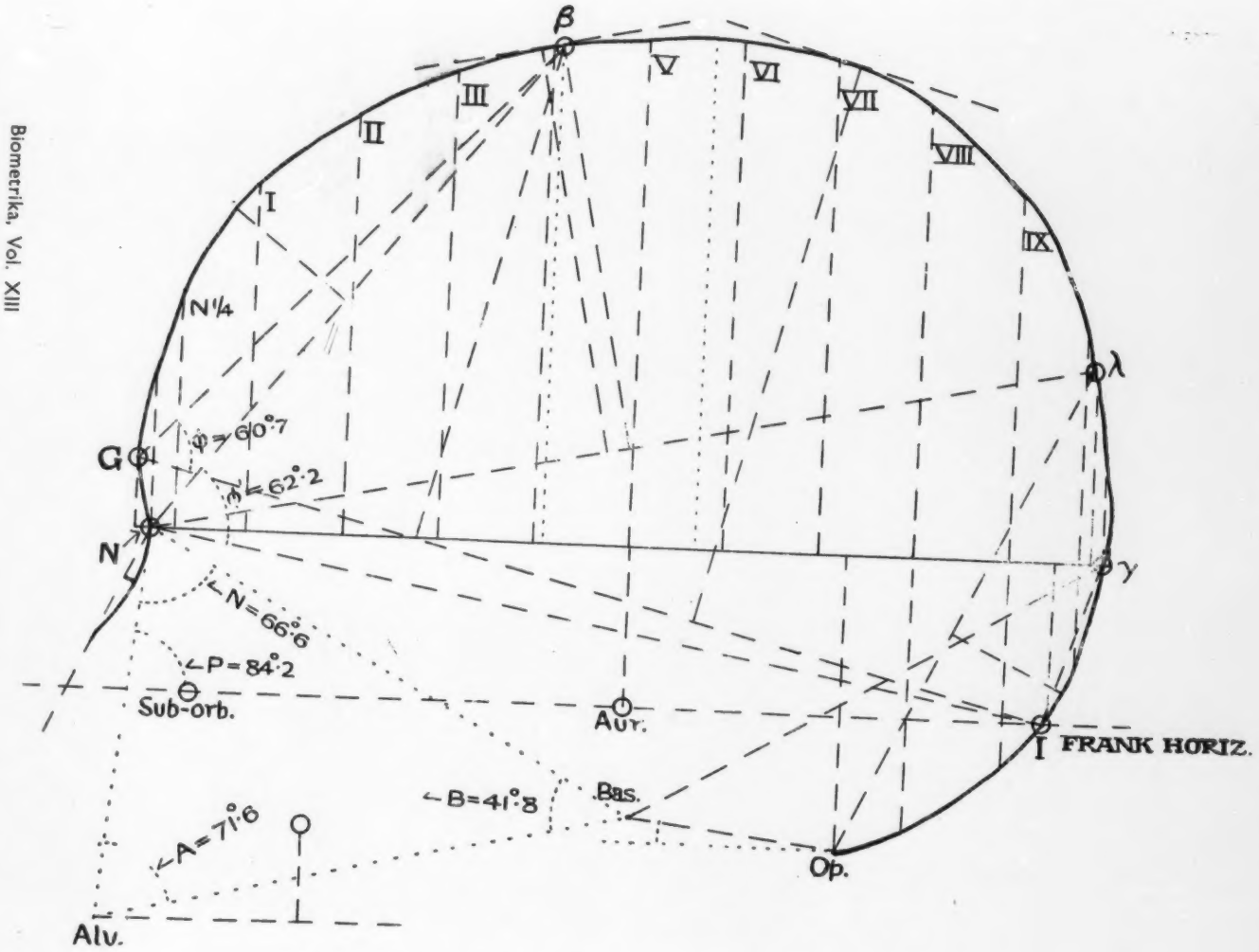


FIG. XVII. Burmese Crania. Type B (? Hybrids). ♀. Sagittal Contour. (17 crania).

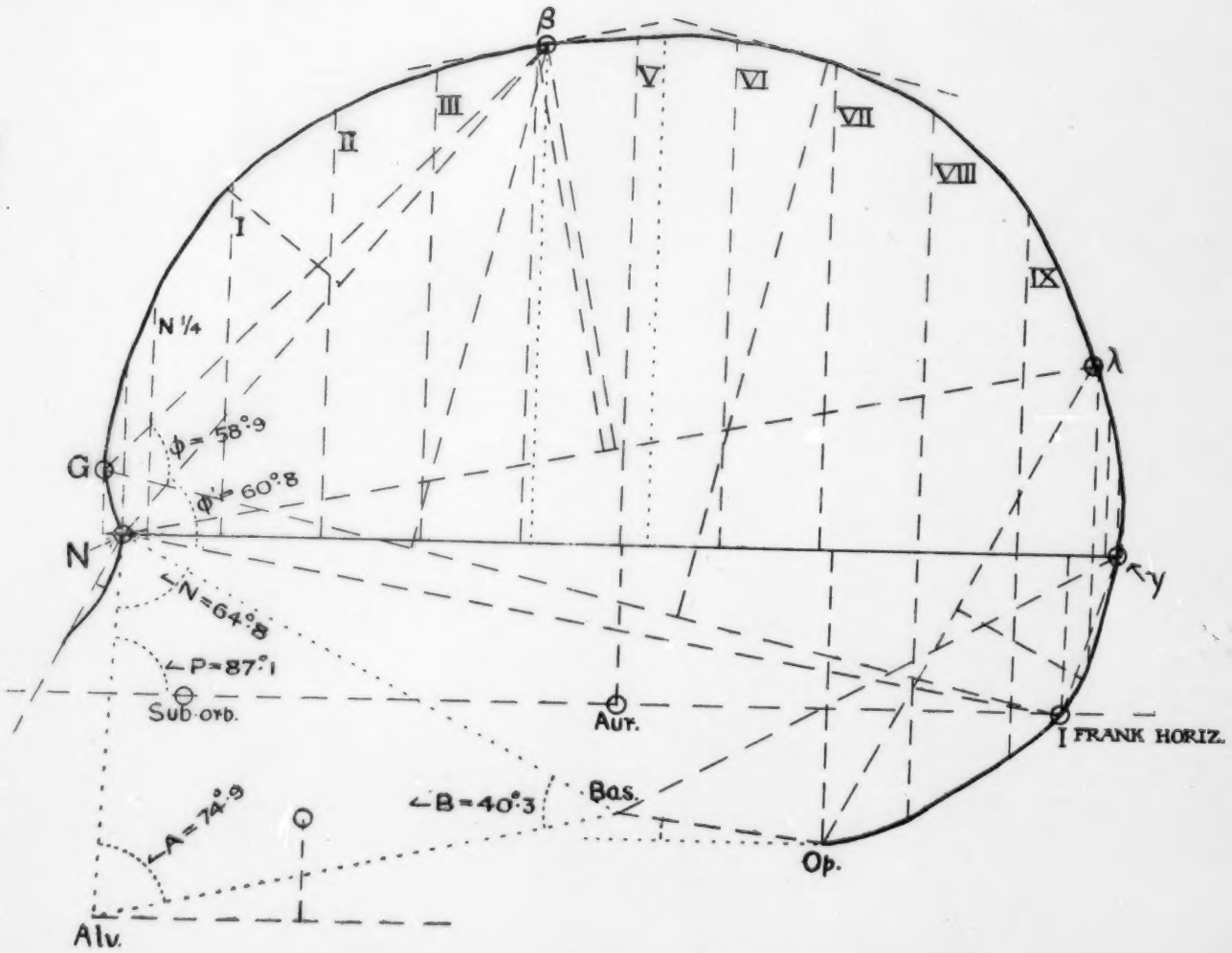


FIG. XVIII. Burmese Crania. Type C (? Karens). ♀. Sagittal Contour. (18 crania.)